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On second-order and fourth-order moments of jointly distributed random matrices: a survey

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Abstract

The study is concerned with second-order and fourth-order moments of jointly distributed random matrices. When distributional properties are required, normality is adopted. Some of the results can also be applied to elliptical or Wishart distributions. The developments are entirely algebraic. Full use is made of the Kronecker product, (repeated) vectorization, commutation matrices and related items. There are a few references in the main text but many additional references to other work, in which the same or kindred results (obtained by other methods, procedures or concepts) can be found, have been included in the References. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

This study is concerned with second-order and fourth-order moments of jointly (if that is expedient: normally) distributed random matrices X and Y. We assume that X and Y have identical row orders, n say. Hence $Z := (X, Y)$ is a feasible definition. Several (co)variance specifications for X and Y will be considered.

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Generally, we write the variance of vec Z as

$$\mathscr{D}(\mathrm{vec}\,Z)=\begin{bmatrix}\Omega_{XX} & \Omega_{XY}\\ \Omega_{YX} & \Omega_{YY}\end{bmatrix},$$

where $\Omega_{XY}:=C(\mathrm{vec}\,X,\ \mathrm{vec}\,Y)$, the covariance of vec X and vec Y, $\Omega_{XX}:=\mathscr{D}(\mathrm{vec}$ $X)$ and $\Omega_{YY}:=\mathscr{D}(\mathrm{vec}\,Y)$.

This variance can subsequently be specialized to

$$\mathscr{D}(\mathrm{vec}\,Z)=\Phi\otimes I_n,$$

where

$$\Phi=\begin{bmatrix}\Phi_{XX} & \Phi_{XY}\\ \Phi_{YX} & \Phi_{YY}\end{bmatrix}.$$

If we write $X'=(x_1,\ldots,x_n)$ and $Y'=(y_1,\ldots,y_n)$, then this specification expresses homoskedasticity of the vector $z_k:=\left[\begin{smallmatrix}x_k\\ y_k\end{smallmatrix}\right]$ $(k=1,\ldots,n)$ and zero-correlation between z_k and z_ℓ $(k\neq\ell;\ k,\ell=1,\ldots,n)$. Expected values of both $X\otimes X$ and $X'\otimes X$ follow straightforwardly for both $\mathscr{D}(\mathrm{vec}\,X)=\Omega_{XX}$ and $\mathscr{D}(\mathrm{vec}\,X)=\Phi_{XX}\otimes I_n$, but also for the intermediate specification $\mathscr{D}(\mathrm{vec}\,X)=V\otimes U$, with psd$(n\times n)U$ and $(p\times p)V$. The latter specification can easily be extended to $\mathscr{D}(\mathrm{vec}\,X)=\sum_{ij}(E_{ij}\otimes\Psi_{ij})$, where E_{ij} $(i,j=1,2,\ldots,p)$ is a basis matrix. This specification could be called general. It is occasionally used.

In Section 2, basic algebraic definitions and properties are being reviewed. These cover, inter alia, the Kronecker product, (repeated) vectorization, commutation matrices with two or more indexes and related items.

In Section 3, basic statistical properties are being reported. We mention expected values and variances of the Kronecker product $x\otimes y$, where x and y are jointly (if that is expedient: normally) distributed random vectors, and the expected value of the quadruple Kronecker product $x\otimes x\otimes x\otimes x$.

Section 4 starts with expected values of the Kronecker products $X\otimes Y$ and $X'\otimes Y$, and then covers the expressions $X\otimes X$ and $X'\otimes X$ as special cases. All variance specifications are considered.

Section 5 treats the variances of the same Kronecker products as in the preceding section, now under the condition of normality. All variance specifications are considered.

In Section 6, we examine expected values of matrix bilinear forms $X'AY,\ XAY'$ and XAY with fixed (generic) A. The expected values of $Y'AX,\ YAX',\ YAX$ and $Y'AX'$ follow immediately, by substitution or transposition.

Finally, the matrix quadratic forms $X'AX,\ XAX'$ and XAX are being treated. Some variance specifications are elaborated.

Section 7 is concerned with the covariance of two matrix quadratic forms in a normally distributed matrix variable X, with variance $\mathscr{D}(\mathrm{vec}\,X)=V\otimes U$, which is subsequently specialized to $\Phi_{XX}\otimes I_n$.

In Section 8, we consider the fourth-order moment $E(X\otimes X\otimes X\otimes X)$ for a normally distributed X. We consider $\mathscr{D}(\mathrm{vec}\,X)=V\otimes U$ and the generalization inspired by it.

Section 9 is dedicated to the expression $X'AXDX'B'X$, with fixed matrices A, D and B. One could call this a matrix bilinear form in two matrix quadratic forms. Our objective is to find its expected value under normality.

We consider $\mathscr{D}(\operatorname{vec} X) = V \otimes U$, and its specialization $\Phi_{XX} \otimes I_n$. Although extensions to other expressions like $X'AXDX'BX$ are feasible, these are not examined here. It becomes clear from the approach how this can be done.

In Section 10, we turn to the topic of the covariance of two matrix bilinear forms in jointly (normally) distributed random matrices X and Y. This is elaborated solely for the variance

$$\Omega = \begin{bmatrix} \Omega_{XX} & \Omega_{XY} \\ \Omega_{YX} & \Omega_{YY} \end{bmatrix}.$$

2. Basic algebraic definitions and properties

In the following, a_{ij} will denote the ijth element of a matrix A, $A_{i.}$ will denote its ith row and $A_{.j}$ its jth column. In certain cases, $(A)_{ij}$ will be used to denote the ijth element of A. All vectors and matrices are real.

The set of $(m \times 1)$ vectors e_i $(i = 1, \ldots, m)$, whose ith element is 1 whereas all other elements are 0, forms a canonical basis for $\mathbb{R}^m$.

Similarly, the set of $(n \times 1)$ vectors e_j $(j = 1, \ldots, n)$, whose jth element is 1 whereas all other elements are 0, forms a canonical basis for $\mathbb{R}^n$. Further, $E_{ij} = e_i e_j'$ is a canonical basis for $\mathbb{R}^{mn}$.

Thus, for any $(m \times n)$ matrix A we can write

$$a_{ij} = e_i' A e_j, \quad A_{i.} = e_i' A, \quad A_{.j} = A e_j,$$

$$A = \sum_{ij} a_{ij} E_{ij} = \sum_j A_{.j} e_j' = \sum_i e_i A_{i.}$$

Clearly,

$$\sum_i E_{ii} = \sum_i e_i e_i' = I_m \quad \text{and} \quad \sum_j E_{jj} = \sum_j e_j e_j' = I_n.$$

Let $A = [a_{ij}]$ be an $(m \times n)$ matrix and B a $(p \times q)$ matrix. Then the Kronecker product $A \otimes B$ is defined as the $(mp \times nq)$ matrix

$$A \otimes B := [a_{ij} B].$$

For basic properties of the Kronecker product we refer to the literature, e.g., [17]. Related to the $(m \times n)$ matrix A is the $(mn \times 1)$ vector

$$\operatorname{vec} A := \begin{bmatrix} A_{.1} \\ \vdots \\ A_{.n} \end{bmatrix} = \sum_{j=1}^{n} (e_j \otimes A_{.j}).$$

The main properties to be used in this paper are:

$$\operatorname{vec} a' = \operatorname{vec} a = a \quad \text{for any column vector } a,$$

$$\operatorname{vec} ab' = b \otimes a \quad \text{for any pair of column vectors } a \text{ and } b,$$

$$\begin{aligned}\operatorname{vec} ABC &= (C' \otimes A)\operatorname{vec} B \\ &= \operatorname{vec}\{(C' \otimes A)\operatorname{vec} B\} \\ &= (\operatorname{vec} B \otimes I_{mp})'\operatorname{vec}(C' \otimes A) \\ &\qquad \text{for compatible matrices } A,\ B \text{ and } C, \\ &\qquad \text{where } mp \text{ is the row order of } C' \otimes A,\end{aligned} \tag{2.1}$$

$$\operatorname{tr} A'B = (\operatorname{vec} A)'\operatorname{vec} B \quad \text{for compatible matrices } A \text{ and } B,$$

$$\begin{aligned}\sum_{ij}(E_{ij} \otimes E_{ij}) &= (\operatorname{vec} I_m)(\operatorname{vec} I_n)' \\ &\qquad \text{for the } (m \times n) \text{ basis matrix } E_{ij} \\ &\qquad (i = 1, \ldots, m;\ j = 1, \ldots, n).\end{aligned} \tag{2.2}$$

The so-called commutation matrix K_{mn} is algebraically defined as the $(mn \times mn)$ matrix

$$K_{mn} := \sum_{ij}(E_{ij} \otimes E'_{ij}) \quad (i = 1, \ldots, m;\ j = 1, \ldots, n).$$

Occasionally we write $K_{m,n}$.

We quote the following properties:

$$K_{mn}\operatorname{vec} A = \operatorname{vec} A' \quad \text{for any } (m \times n) \text{ matrix } A, \tag{2.3}$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD \quad \text{for compatible matrices } A,\ B,\ C \text{ and } D, \tag{2.4}$$

$$K'_{mn} = K_{mn}^{-1} = K_{nm} \quad (\text{since } K_{mn} \text{ is a permutation matrix and hence is orthogonal}), \tag{2.5}$$

$$K_{m1} = K_{1m} = I_m, \tag{2.6}$$

$$K_{pm}(A \otimes B)K_{nq} = B \otimes A \quad \text{for } (m \times n) \text{ matrix } A \text{ and } (p \times q) \text{ matrix } B, \tag{2.7}$$

$$\operatorname{vec}(A \otimes B) = (I_n \otimes K_{qm} \otimes I_p)(\operatorname{vec} A \otimes \operatorname{vec} B) \quad \text{for } A \text{ and } B \text{ as defined in (2.7)}, \tag{2.8}$$

$$K_{rs,m} = (I_r \otimes K_{sm})(K_{rm} \otimes I_s), \quad r \text{ and } s \text{ being interchangeable,} \tag{2.9}$$

$$K_{q,mnp} = (K_{qm} \otimes I_{np})(I_m \otimes K_{qn} \otimes I_p)(I_{mn} \otimes K_{qp}), \quad m,\ n \text{ and } p \text{ being interchangeable,} \tag{2.10}$$

$$\begin{aligned} K_{pq,mn} = {} & (I_p \otimes K_{qm} \otimes I_n)(K_{pm} \otimes K_{qn})(I_m \otimes K_{pn} \otimes I_q), \\ & p \text{ and } q \text{ being interchangeable,} \\ & m \text{ and } n \text{ being interchangeable.} \end{aligned} \tag{2.11}$$

For the last four properties, see [25].

Four additional useful properties are as follows:

$$(I_n \otimes K_{mm} \otimes I_n)\operatorname{vec} K_{mn} = \operatorname{vec} K_{mn}. \tag{2.12}$$

Proof.

$$\begin{aligned} & (I_n \otimes K_{mm} \otimes I_n) \operatorname{vec} K_{mn} \\ & \quad = \sum_{ij} (I_n \otimes K_{mm} \otimes I_n) \operatorname{vec}(E_{ij} \otimes E'_{ij}) \\ & \quad = \sum_{ij} (\operatorname{vec} E_{ij} \otimes \operatorname{vec} E'_{ij}) \\ & \quad = \sum_{ij} \{(e_j \otimes e_i) \otimes (e_i \otimes e_j)\} \\ & \quad = \sum_{ij} \operatorname{vec}\{(e_i \otimes e_j)(e_j \otimes e_i)'\} \\ & \quad = \operatorname{vec} \sum_{ij} (e_i e'_j \otimes e_j e'_i) \\ & \quad = \operatorname{vec} \sum_{ij} (E_{ij} \otimes E'_{ij}) \\ & \quad = \operatorname{vec} K_{mn}. \end{aligned}$$

We used $E_{ij} = e_i e'_j$, e_i and e_j being $(m \times 1)$ and $(n \times 1)$ basis vectors, respectively. □

$$C_2^n C_3^n C_2^n = C_3^n C_2^n C_3^n, \qquad C_3^n C_2^n = I_n \otimes K_{n^2,n}, \tag{2.13}$$

where

$$C_2^n := I_n \otimes K_{nn} \otimes I_n, \qquad C_3^n := I_{n^2} \otimes K_{nn}.$$

Proof. Consider

$$\begin{aligned}
&(K_{nn}\otimes I_n)(I_n\otimes K_{nn})(K_{nn}\otimes I_n)\\
&\quad=(K_{nn}\otimes I_n)K_{n^2,n}\\
&\quad=K_{n^2,n}(I_n\otimes K_{nn})\\
&\quad=(I_n\otimes K_{nn})(K_{nn}\otimes I_n)(I_n\otimes K_{nn})\quad\text{(by (2.9))}.
\end{aligned}$$

Hence

$$\begin{aligned}
C_2^nC_3^nC_2^n&=I_n\otimes(K_{nn}\otimes I_n)(I_n\otimes K_{nn})(K_{nn}\otimes I_n)\\
&=I_n\otimes(I_n\otimes K_{nn})(K_{nn}\otimes I_n)(I_n\otimes K_{nn})\\
&=C_3^nC_2^nC_3^n.
\end{aligned}$$

Further,

$$\begin{aligned}
C_3^nC_2^n&=I_n\otimes(I_n\otimes K_{nn})(K_{nn}\otimes I_n)\\
&=I_n\otimes K_{n^2,n}\quad\text{(by (2.9))}.\qquad\square
\end{aligned}$$

$$N'AUB'M=(\operatorname{vec}A\otimes I_q)'(U\otimes\tilde{\nu}\tilde{\mu}')(\operatorname{vec}B\otimes I_p)$$

for any $(n\times p)$ *matrix* M, $(n\times q)$ *matrix* N *and* $(n\times n)$ *matrices* $A,\ B,\ U$ *with further*

$$\tilde{\mu}:=\operatorname{vec}M'\quad\textit{and}\quad\tilde{\nu}:=\operatorname{vec}N'.\tag{2.14}$$

Proof. Use the decomposition $ABC=\sum_{ij}b_{ij}A_{.i}C_{j.}$ Then

$$\begin{aligned}
N'AUB'M&=\sum_{ij}u_{ij}(N'A)_{.i}(B'M)_{j.}\\
&=\sum_{ij}u_{ij}N'A_{.i}(M'B_{.j})'\\
&=\sum_{ij}u_{ij}(\operatorname{vec}N'A_{.i})(\operatorname{vec}M'B_{.j})'\\
&=\sum_{ij}u_{ij}(A_{.i}\otimes I_q)'\tilde{\nu}\tilde{\mu}'(B_{.j}\otimes I_p)\\
&=\sum_{ij}(A_{.i}\otimes I_q)'u_{ij}\tilde{\nu}\tilde{\mu}'(B_{.j}\otimes I_p)\\
&=(\operatorname{vec}A\otimes I_q)'(U\otimes\tilde{\nu}\tilde{\mu}')(\operatorname{vec}B\otimes I_p).\qquad\square
\end{aligned}$$

$$N'A'UBM=(I_q\otimes\operatorname{vec}A)'(\nu\mu'\otimes U)(I_p\otimes\operatorname{vec}B)\tag{2.15}$$

for $M,\ N,\ A,\ U$ *and* B *as defined in* (2.14) *and* $\mu:=\operatorname{vec}M$, $\nu:=\operatorname{vec}N$.

Proof. By (2.14)

$$\begin{aligned}N'A'UBM &= (\operatorname{vec} A' \otimes I_q)'(U \otimes \tilde{\nu}\tilde{\mu}')(\operatorname{vec} B' \otimes I_p)\\ &= (\operatorname{vec} A \otimes I_q)'(K_{nn} \otimes I_q)K_{n,nq}(\tilde{\nu}\tilde{\mu}' \otimes U)\\ &\quad \cdot K_{np,n}(K_{nn} \otimes I_p)(\operatorname{vec} B \otimes I_p)\\ &= (\operatorname{vec} A \otimes I_q)'(I_n \otimes K_{nq})(\tilde{\nu}\tilde{\mu}' \otimes U)(I_n \otimes K_{pn})(\operatorname{vec} B \otimes I_p)\\ &= (\operatorname{vec} A \otimes I_q)'(I_n \otimes K_{nq})(K_{nq} \otimes I_n)(\nu\mu' \otimes U)(K_{pn} \otimes I_n)\\ &\quad \cdot (I_n \otimes K_{pn})(\operatorname{vec} B \otimes I_p)\\ &= (\operatorname{vec} A \otimes I_q)'K_{nn,q}(\nu\mu' \otimes U)K_{p,nn}(\operatorname{vec} B \otimes I_p)\\ &= (I_q \otimes \operatorname{vec} A)'(\nu\mu' \otimes U)(I_p \otimes \operatorname{vec} B),\end{aligned}$$

by virtue of (2.3), (2.7) and (2.9). □

3. Basic statistical properties

3.1.

Consider jointly normally distributed random vectors $(n \times 1)x$ and $(m \times 1)y$. Define

$$z := \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $z \sim N_{n+m}(\mu, \Omega)$, where

$$E(z) = \mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} = \begin{bmatrix} E(x) \\ E(y) \end{bmatrix}$$

and

$$\mathscr{D}(z) = \Omega = \begin{bmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{bmatrix} = \begin{bmatrix} \mathscr{D}(x) & C(x, y) \\ C(y, x) & \mathscr{D}(y) \end{bmatrix},$$

$C(x, y)$ denoting the covariance matrix of x and y:

$$C(x, y) = E(x - \mu_x)(y - \mu_y)' = E(xy') - \mu_x\mu_y'.$$

The matrix Ω is positive semi-definite. The following properties hold:

$$E(x \otimes y) = \operatorname{vec} \Omega_{yx} + \mu_x \otimes \mu_y, \tag{3.1}$$

$$E(x \otimes x) = \operatorname{vec} \Omega_{xx} + \mu_x \otimes \mu_x, \tag{3.2}$$

$$\begin{aligned}\mathscr{D}(x \otimes y) &= \Omega_{xx} \otimes \Omega_{yy} + \Omega_{xx} \otimes \mu_y\mu_y' + \mu_x\mu_x' \otimes \Omega_{yy}\\ &\quad + K_{nm}(\Omega_{yx} \otimes \Omega_{xy} + \Omega_{yx} \otimes \mu_x\mu_y' + \mu_y\mu_x' \otimes \Omega_{xy}),\end{aligned} \tag{3.3}$$

$$\mathscr{D}(x \otimes x) = (I_{n^2} + K_{nn})(\Omega_{xx} \otimes \Omega_{xx} + \Omega_{xx} \otimes \mu_x \mu_x' + \mu_x \mu_x' \otimes \Omega_{xx}). \quad (3.4)$$

For a derivation, see [16]. The generalization from positive definite to positive semi-definite Ω is straightforward. See e.g., [22].

Note. Properties (3.1) and (3.2) hold for any distribution.

When we drop normality and introduce stochastic independence for x and y, we get:

$$\mathscr{D}(x \otimes y) = \Omega_{xx} \otimes \Omega_{yy} + \Omega_{xx} \otimes \mu_y \mu_y' + \mu_x \mu_x' \otimes \Omega_{yy},$$

i.e. the first part of (3.3).

Proof.

$$\begin{aligned}\mathscr{D}(x \otimes y) &= E[(x \otimes y)(x \otimes y)'] - [E(x \otimes y)][E(x \otimes y)]' \\ &= E(xx' \otimes yy') - [(Ex) \otimes (Ey)][(Ex) \otimes (Ey)]' \\ &= E(xx') \otimes E(yy') - (\mu_x \otimes \mu_y)(\mu_x \otimes \mu_y)' \\ &= (\Omega_{xx} + \mu_x \mu_x') \otimes (\Omega_{yy} + \mu_y \mu_y') - \mu_x \mu_x' \otimes \mu_y \mu_y' \\ &= \Omega_{xx} \otimes \Omega_{yy} + \Omega_{xx} \otimes \mu_y \mu_y' + \mu_x \mu_x' \otimes \Omega_{yy}. \qquad \square\end{aligned}$$

3.2.

A basic fourth-order result is the following:

For $x \sim N_n(\mu, \Omega)$:

$$\begin{aligned}E(x \otimes x \otimes x \otimes x) = &\operatorname{vec}[(I_{n^2} + K_{nn})(\Omega \otimes \Omega + \Omega \otimes \mu\mu' + \mu\mu' \otimes \Omega)] \\ &+ \operatorname{vec}(\Omega + \mu\mu') \otimes \operatorname{vec}(\Omega + \mu\mu'). \end{aligned} \quad (3.5)$$

Proof. Write $x \otimes x \otimes x \otimes x = \operatorname{vec} xx' \otimes \operatorname{vec} xx' = \operatorname{vec}[(\operatorname{vec} xx')(\operatorname{vec} xx')']$. Hence

$$\begin{aligned}E(x \otimes x \otimes x \otimes x) &= \operatorname{vec} E[(\operatorname{vec} xx')(\operatorname{vec} xx')'] \\ &= \operatorname{vec}[D(\operatorname{vec} xx')] + \operatorname{vec}[(E\operatorname{vec} xx')(E\operatorname{vec} xx')'] \\ &= \operatorname{vec}[(I_{n^2} + K_{nn})(\Omega \otimes \Omega + \Omega \otimes \mu\mu' + \mu\mu' \otimes \Omega)] \\ &\quad + \operatorname{vec}[\{\operatorname{vec}(\Omega + \mu\mu')\}\{(\operatorname{vec}(\Omega + \mu\mu')\}'] \\ &= \operatorname{vec}[(I_{n^2} + K_{nn})(\Omega \otimes \Omega + \Omega \otimes \mu\mu' + \mu\mu' \otimes \Omega)] \\ &\quad + \operatorname{vec}(\Omega + \mu\mu') \otimes \operatorname{vec}(\Omega + \mu\mu'),\end{aligned}$$

by virtue of (3.2) and (3.4). $\square$

4. Second-order moments for a pair of jointly distributed random matrices

Consider the random matrix $Z = (X, Y)$ with $(n \times p)X$ and $(n \times q)Y$. Let $E(Z) = (M, N)$ and

$$\mathscr{D}(\operatorname{vec} Z) = \Omega = \begin{bmatrix} \Omega_{XX} & \Omega_{XY} \\ \Omega_{YX} & \Omega_{YY} \end{bmatrix}.$$

Then:

$$E(X \otimes Y) = \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{YX}(I_p \otimes E'_{ij}) + M \otimes N, \tag{4.1}$$

where E_{ij} is a $(q \times n)$ basis matrix $(i = 1, \ldots, q;\ j = 1, \ldots, n)$.

Proof.

$$\begin{aligned}
\operatorname{vec} E(X \otimes Y) &= (I_p \otimes K_{qn} \otimes I_n)E(\operatorname{vec} X \otimes \operatorname{vec} Y) \\
&= (I_p \otimes K_{qn} \otimes I_n)(\operatorname{vec} \Omega_{YX} + \operatorname{vec} M \otimes \operatorname{vec} N) \\
&= \sum_{ij} (I_p \otimes E_{ij} \otimes E'_{ij} \otimes I_n)\operatorname{vec} \Omega_{YX} + \operatorname{vec}(M \otimes N) \\
&= \operatorname{vec} \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{YX}(I_p \otimes E'_{ij}) + \operatorname{vec}(M \otimes N),
\end{aligned}$$

by virtue of (2.8), (2.1), (3.1) and the definition of K_{qn}. □

$$E(X' \otimes Y) = \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{YX}(E'_{ij} \otimes I_n)K_{qn} + M' \otimes N, \tag{4.2}$$

where E_{ij} is a $(q \times p)$ basis matrix $(i = 1, \ldots, q;\ j = 1, \ldots, p)$.

Proof.

$$\begin{aligned}
\operatorname{vec} E(X' \otimes Y) &= (I_n \otimes K_{qp} \otimes I_n)E(\operatorname{vec} X' \otimes \operatorname{vec} Y) \\
&= (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq})E(\operatorname{vec} X \otimes \operatorname{vec} Y) \\
&= (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq})(\operatorname{vec} \Omega_{YX} + \operatorname{vec} M \otimes \operatorname{vec} N) \\
&= (I_n \otimes K_{qp} \otimes I_n)(\operatorname{vec} \Omega_{YX} K_{pn} + \operatorname{vec} M' \otimes \operatorname{vec} N) \\
&= \sum_{ij} (I_n \otimes E_{ij} \otimes E'_{ij} \otimes I_n)\operatorname{vec} \Omega_{YX} K_{pn} + \operatorname{vec}(M' \otimes N) \\
&= \operatorname{vec} \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{YX} K_{pn}(I_n \otimes E'_{ij}) + \operatorname{vec}(M' \otimes N) \\
&= \operatorname{vec} \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{YX}(E'_{ij} \otimes I_n)K_{qn} + \operatorname{vec}(M' \otimes N),
\end{aligned}$$

by virtue of (2.8), (2.3), (2.1), (2.5), (2.7) and (3.1). □

Corollaries.

$$E(X \otimes X) = \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{XX}(I_p \otimes E'_{ij}) + M \otimes M, \tag{4.3}$$

where E_{ij} is a $(p \times n)$ basis matrix $(i = 1, \ldots, p;\ j = 1, \ldots, n)$.

$$E(X' \otimes X) = \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{XX}(E'_{ij} \otimes I_n)K_{pn} + M' \otimes M, \tag{4.4}$$

where E_{ij} is a $(p \times p)$ basis matrix $(i, j = 1, \ldots, p)$.

For the special case: $\Omega_{XX} = V \otimes U$, with psd$(n \times n)U$ and $(p \times p)V$, we find

$$E(X \otimes X) = (\operatorname{vec} U)(\operatorname{vec} V)' + M \otimes M. \tag{4.5}$$

Proof. Using (4.3) we find that

$$\begin{aligned}
&\sum_{ij} (E'_{ij} \otimes I_n)\Omega_{XX}(I_p \otimes E'_{ij}) \\
&\quad = \sum_{ij} (E'_{ij} \otimes I_n)(V \otimes U)(I_p(E'_{ij}) \\
&\quad = \sum_{ij} (E'_{ij}V \otimes UE'_{ij}) \\
&\quad = \sum_{ij} (e_j V_{i.} \otimes U_{.j} e'_i) \\
&\quad = \sum_{j} (e_j \otimes U_{.j}) \sum_{i} (V_{i.} \otimes e'_i) \\
&\quad = \left(\sum_{j} \operatorname{vec} U_{.j} e'_j \right) \left(\sum_{i} \operatorname{vec} e_i V_{i.} \right)' \\
&\quad = (\operatorname{vec} U)(\operatorname{vec} V)'. \qquad \square
\end{aligned}$$

$$E(X' \otimes X) = K_{pn}(U \otimes V) + M' \otimes M. \tag{4.6}$$

Proof. Using (4.4) we get

$$\begin{aligned}
&\sum_{ij} (E'_{ij} \otimes I_n)\Omega_{XX}(E'_{ij} \otimes I_n)K_{pn} \\
&\quad = \sum_{ij} (E'_{ij} \otimes I_n)(V \otimes U)(E'_{ij} \otimes I_n)K_{pn}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{ij} (E'_{ij} V E'_{ij} \otimes U) K_{pn} \\
&= \sum_{ij} (v_{ij} E'_{ij} \otimes U) K_{pn} \\
&= (V' \otimes U) K_{pn} \\
&= (V \otimes U) K_{pn} = K_{pn}(U \otimes V). \qquad \square
\end{aligned}
$$

For the special case: $\Omega = \Phi \otimes I_n$, where

$$
\Phi = \begin{bmatrix} \Phi_{XX} & \Phi_{XY} \\ \Phi_{YX} & \Phi_{YY} \end{bmatrix},
$$

we get

$$
E(X \otimes Y) = (\mathrm{vec}\, I_n)(\mathrm{vec}\, \Phi_{YX})' + M \otimes N. \tag{4.7}
$$

Proof. Using (4.1) we get

$$
\begin{aligned}
&\sum_{ij} (E'_{ij} \otimes I_n) \Omega_{YX} (I_p \otimes E'_{ij}) \\
&\quad = \sum_{ij} (E'_{ij} \otimes I_n)(\Phi_{YX} \otimes I_n)(I_p \otimes E'_{ij}) \\
&\quad = \sum_{ij} (E'_{ij} \Phi_{YX} \otimes E'_{ij}) \\
&\quad = \sum_{ij} (E'_{ij} \otimes E'_{ij})(\Phi_{YX} \otimes I_q) \\
&\quad = (\mathrm{vec}\, I_n)(\mathrm{vec}\, I_q)'(\Phi_{YX} \otimes I_q) \\
&\quad = (\mathrm{vec}\, I_n)(\mathrm{vec}\, \Phi_{YX})' \quad \text{(by (2.2) and (2.1))}. \qquad \square
\end{aligned}
$$

$$
E(X' \otimes Y) = K_{pn}(I_n \otimes \Phi_{XY}) + M' \otimes N. \tag{4.8}
$$

Proof. Using (4.2) we get

$$
\begin{aligned}
&\sum_{ij} (E'_{ij} \otimes I_n) \Omega_{YX} (E'_{ij} \otimes I_n) K_{qn} \\
&\quad = \sum_{ij} (E'_{ij} \otimes I_n)(\Phi_{YX} \otimes I_n)(E'_{ij} \otimes I_n) K_{qn} \\
&\quad = \sum_{ij} (E'_{ij} \Phi_{YX} E'_{ij} \otimes I_n) K_{qn}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ \sum_{ij} (\Phi_{YX})_{ij} E'_{ij} \otimes I_n \right\} K_{qn} \\
&= (\Phi'_{YX} \otimes I_n) K_{qn} = (\Phi_{XY} \otimes I_n) K_{qn} \\
&= K_{pn}(I_n \otimes \Phi_{XY}). \qquad \square
\end{aligned}
$$

Corollaries of (4.7) *and* (4.8):

$$
E(X \otimes X) = (\operatorname{vec} I_n)(\operatorname{vec} \Phi_{XX})' + M \otimes M, \tag{4.9}
$$

$$
E(X' \otimes X) = K_{pn}(I_n \otimes \Phi_{XX}) + M' \otimes M. \tag{4.10}
$$

Remarks. 1. Corollaries (4.9) and (4.10) also follow as special subcases of (4.5) and (4.6) for $U = I_n$ and $V = \Phi_{XX}$.

2. See for a different method to prove (4.5) and (4.6) [26]. Basic is, e.g., the identity

$$
\begin{aligned}
\operatorname{tr} A'X'PX &= (\operatorname{vec} P)'(X \otimes X)\operatorname{vec} A \\
&= \operatorname{tr}(A \otimes P)(\operatorname{vec} X)(\operatorname{vec} X)'.
\end{aligned}
$$

Its expected value is

$$
\begin{aligned}
(\operatorname{vec} P)'E(X \otimes X)\operatorname{vec} A &= \operatorname{tr}(A \otimes P)[V \otimes U + (\operatorname{vec} M)(\operatorname{vec} M)'] \\
&= (\operatorname{tr} P'U)(\operatorname{tr} VA) + (\operatorname{vec} P)'(M \otimes M)\operatorname{vec} A \\
&= (\operatorname{vec} P)'(\operatorname{vec} U)(\operatorname{vec} V)'\operatorname{vec} A \\
&\quad + (\operatorname{vec} P)'(M \otimes M)\operatorname{vec} A,
\end{aligned}
$$

which holds for any A and P.

Hence,

$$
E(X \otimes X) = (\operatorname{vec} U)(\operatorname{vec} V)' + M \otimes M.
$$

This procedure was employed by the authors to derive $E(X'PX)$, which followed from the equivalent identity

$$
E(\operatorname{vec} A)'\operatorname{vec} X'PX = (\operatorname{tr} PU)(\operatorname{vec} A)'\operatorname{vec} V + (\operatorname{vec} A)'\operatorname{vec} M'PM.
$$

3. An interesting application of (4.3) concerns $S \sim W_p(\Sigma, n)$, the central Wishart distribution. Then

$$
E(S \otimes S) = n(\operatorname{vec} \Sigma)(\operatorname{vec} \Sigma)' + n(nI_{p^2} + K_{pp})(\Sigma \otimes \Sigma).
$$

See e.g., [12, Lemma B7] for $\Sigma = I_p$, in scalar form. Extension to the noncentral case is straightforward.

4. The specification $\Omega_{XX} = V \otimes U$ is not restrictive. Suppose we have

$$
\mathscr{D}(\operatorname{vec} X) = \Omega_{XX} = [\Psi_{ij}] \quad (i, j = 1, \ldots, p).
$$

We then get from (4.5) by the decomposition $\Omega_{XX} = \sum_{ij}(E_{ij} \otimes \Psi_{ij})$, with E_{ij} a $(p \times p)$ basis matrix:

$$E(X \otimes X) = \sum_{ij}(\text{vec}\,\Psi_{ij})(\text{vec}\,E_{ij})' + M \otimes M. \tag{4.11}$$

From (4.6) we get

$$E(X' \otimes X) = K_{pn}\sum_{ij}(\Psi_{ij} \otimes E'_{ij}) + M' \otimes M. \tag{4.12}$$

Expressions (4.11) and (4.12), of course, tally with (4.3) and (4.4), respectively.

Lemma 1. *Expressions* (4.3) *and* (4.11) *are identical.*

Lemma 2. *Expressions* (4.4) *and* (4.12) *are identical.*

Proof of Lemma 1. Consider

$$\begin{aligned}
&\sum_{ij}(E'_{ij} \otimes I_n)\Omega_{XX}(I_p \otimes E'_{ij}) \quad (i = 1, \ldots, p,\ \ j = 1, \ldots, n)\\
&\quad = \sum_{ij}(E'_{ij} \otimes I_n)\sum_{k\ell}(E_{k\ell} \otimes \Psi_{k\ell})(I_p \otimes E'_{ij}) \quad (k, \ell = 1, \ldots, p)\\
&\quad = \sum_{ijk\ell}(E'_{ij}E_{k\ell} \otimes \Psi_{k\ell}E'_{ij})\\
&\quad = \sum_{ijk\ell}(e_je'_ie_ke'_\ell \otimes \Psi_{k\ell}E'_{ij})\\
&\quad = \sum_{ij\ell}(e_je'_\ell \otimes (\Psi_{i\ell})_{.j}e'_i)\\
&\quad = \sum_{ij\ell}\{e_j \otimes (\Psi_{i\ell})_{.j}\}(e_\ell \otimes e_i)'\\
&\quad = \sum_{i\ell}(\text{vec}\,\Psi_{i\ell})(\text{vec}\,E_{i\ell})'. \qquad \square
\end{aligned}$$

Proof of Lemma 2. Consider

$$\begin{aligned}
&\sum_{ij}(E'_{ij} \otimes I_n)\Omega_{XX}(E'_{ij} \otimes I_n)K_{pn} \quad (i, j = 1, \ldots, p)\\
&\quad = \sum_{ij}(E'_{ij} \otimes I_n)\sum_{k\ell}(E_{k\ell} \otimes \Psi_{k\ell})(E'_{ij} \otimes I_n)K_{pn} \quad (k, \ell = 1, \ldots, p)\\
&\quad = \sum_{ijk\ell}(E'_{ij}E_{k\ell}E_{ij} \otimes \Psi_{k\ell})K_{pn}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{ijk\ell} (e_j e_i' e_k e_\ell' e_j e_i' \otimes \Psi_{k\ell}) K_{pn} \\
&= \sum_{ij} (e_j e_i' \otimes \Psi_{ij}) K_{pn} \\
&= \sum_{ij} (E_{ij}' \otimes \Psi_{ij}) K_{pn} \\
&= K_{pn} \sum_{ij} (\Psi_{ij} \otimes E_{ij}'). \qquad \square
\end{aligned}
$$

5. Variances of $\mathrm{vec}(X \otimes Y)$, $\mathrm{vec}(X' \otimes Y)$, $\mathrm{vec}(X \otimes X)$ and $\mathrm{vec}(X' \otimes X)$ when $\mathrm{vec}\, X$ and $\mathrm{vec}\, Y$ follow a joint normal distribution

5.1.

We consider again the random matrix

$$Z := (X, Y)$$

with $(n \times p)X$ and $(n \times q)Y$, and define $z := \mathrm{vec}\, Z,\ x := \mathrm{vec}\, X,\ y := \mathrm{vec}\, Y$.

Let

$$z \sim N\{\mathrm{vec}(\mu, \nu), \Omega\},$$

where

$$\mu := \mathrm{vec}\, M, \quad \nu := \mathrm{vec}\, N, \quad E(X) = M, \quad E(Y) = N$$

and

$$\Omega = \begin{bmatrix} \Omega_{XX} & \Omega_{XY} \\ \Omega_{YX} & \Omega_{YY} \end{bmatrix} = \begin{bmatrix} \mathscr{D}(x) & C(x, y) \\ C(y, x) & \mathscr{D}(y) \end{bmatrix}.$$

We shall first derive $\mathscr{D}\{\mathrm{vec}(X \otimes Y)\}$ and $\mathscr{D}\{\mathrm{vec}(X' \otimes Y)\}$. This yields for $\mathscr{D}\{\mathrm{vec}(X \otimes Y)\}$ in the general case:

$$
\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X \otimes Y)\} \\
&\quad = (I_p \otimes K_{qn} \otimes I_n)(\Omega_{XX} \otimes \Omega_{YY} + \Omega_{XX} \otimes \nu\nu' + \mu\mu' \otimes \Omega_{YY}) \\
&\qquad \cdot (I_p \otimes K_{nq} \otimes I_n) + (K_{pq} \otimes K_{nn})(I_q \otimes K_{pn} \otimes I_n) \\
&\qquad \cdot (\Omega_{YX} \otimes \Omega_{XY} + \Omega_{YX} \otimes \mu\nu' + \nu\mu' \otimes \Omega_{XY})(I_p \otimes K_{nq} \otimes I_n).
\end{aligned} \tag{5.1}
$$

Proof. We rewrite $\mathrm{vec}(X \otimes Y) = (I_p \otimes K_{qn} \otimes I_n)(x \otimes y)$. Hence

$$
\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X \otimes Y)\} \\
&\quad = (I_p \otimes K_{qn} \otimes I_n)\{\mathscr{D}(x \otimes y)\}(I_p \otimes K_{nq} \otimes I_n) \\
&\quad = (I_p \otimes K_{qn} \otimes I_n)\{\Omega_{XX} \otimes \Omega_{YY} + \Omega_{XX} \otimes \nu\nu' + \mu\mu' \otimes \Omega_{YY} \\
&\qquad + K_{np,nq}(\Omega_{YX} \otimes \Omega_{XY} + \Omega_{YX} \otimes \mu\nu' + \nu\mu' \otimes \Omega_{XY})\}(I_p \otimes K_{nq} \otimes I_n)
\end{aligned}
$$

$$
\begin{aligned}
&= (I_p \otimes K_{qn} \otimes I_n)(\Omega_{XX} \otimes \Omega_{YY} + \Omega_{XX} \otimes \nu\nu' + \mu\mu' \otimes \Omega_{YY}) \\
&\quad \cdot (I_p \otimes K_{nq} \otimes I_n) + (K_{pq} \otimes K_{nn})(I_q \otimes K_{pn} \otimes I_n) \\
&\quad \cdot (\Omega_{YX} \otimes \Omega_{XY} + \Omega_{YX} \otimes \mu\nu' + \nu\mu' \otimes \Omega_{XY})(I_p \otimes K_{nq} \otimes I_n).
\end{aligned}
$$

Use was made of (2.11) and (3.3). □

For $\mathscr{D}\{\mathrm{vec}(X' \otimes Y)\}$, *we get*

$$
\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes Y)\} \\
&\quad = (I_n \otimes K_{qp} \otimes I_n)(\tilde{\Omega}_{XX} \otimes \Omega_{YY} + \tilde{\Omega}_{XX} \otimes \nu\nu' + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{YY}) \\
&\quad\quad \cdot (I_n \otimes K_{pq} \otimes I_n) + (K_{nq} \otimes K_{pn})(I_q \otimes K_{nn} \otimes I_p) \\
&\quad\quad \cdot (\Omega_{YX} K_{pn} \otimes K_{np}\Omega_{XY} + \Omega_{YX} K_{pn} \otimes \tilde{\mu}\nu' + \nu\tilde{\mu}' \otimes K_{np}\Omega_{XY}) \\
&\quad\quad \cdot (I_n \otimes K_{pq} \otimes I_n),
\end{aligned}
\tag{5.2}
$$

where $\tilde{\Omega}_{XX} := K_{np}\Omega_{XX}K_{pn}$ *and* $\tilde{\mu} := \mathrm{vec}\, M'$.

Proof. We rewrite

$$
\begin{aligned}
\mathrm{vec}(X' \otimes Y) &= (I_n \otimes K_{qp} \otimes I_n)(\mathrm{vec}\, X' \otimes \mathrm{vec}\, Y) \\
&= (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq})(x \otimes y).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes Y)\} \\
&\quad = (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq})\{\mathscr{D}(x \otimes y)\}(K_{pn} \otimes I_{nq})(I_n \otimes K_{pq} \otimes I_n) \\
&\quad = (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq})\{\Omega_{XX} \otimes \Omega_{YY} + \Omega_{XX} \otimes \nu\nu' + \mu\mu' \otimes \Omega_{YY} \\
&\quad\quad + K_{np,nq}(\Omega_{YX} \otimes \Omega_{XY} + \Omega_{YX} \otimes \mu\nu' + \nu\mu' \otimes \Omega_{XY})\} \\
&\quad\quad \cdot (K_{pn} \otimes I_{nq})(I_n \otimes K_{pq} \otimes I_n) \\
&\quad = (I_n \otimes K_{qp} \otimes I_n)(K_{np}\Omega_{XX}K_{pn} \otimes \Omega_{YY} + K_{np}\Omega_{XX}K_{pn} \otimes \nu\nu' \\
&\quad\quad + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{YY})(I_n \otimes K_{pq} \otimes I_n) + (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq}) \\
&\quad\quad \cdot K_{np,nq}(\Omega_{YX} \otimes \Omega_{XY} + \Omega_{YX} \otimes \mu\nu' + \nu\mu' \otimes \Omega_{XY}) \\
&\quad\quad \cdot (K_{pn} \otimes I_{nq})(I_n \otimes K_{pq} \otimes I_n) \\
&\quad = (I_n \otimes K_{qp} \otimes I_n)(\tilde{\Omega}_{XX} \otimes \Omega_{YY} + \tilde{\Omega}_{XX} \otimes \nu\nu' + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{YY}) \\
&\quad\quad \cdot (I_n \otimes K_{pq} \otimes I_n) + (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq}) \\
&\quad\quad \cdot K_{np,nq}(\Omega_{YX}K_{pn} \otimes \Omega_{XY} + \Omega_{YX}K_{pn} \otimes \mu\nu' + \nu\tilde{\mu}' \otimes \Omega_{XY}) \\
&\quad\quad \cdot (I_n \otimes K_{pq} \otimes I_n)
\end{aligned}
$$

$$\begin{aligned}
&= (I_n \otimes K_{qp} \otimes I_n)(\tilde{\Omega}_{XX} \otimes \Omega_{YY} + \tilde{\Omega}_{XX} \otimes \nu\nu' + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{YY}) \\
&\quad \cdot(I_n \otimes K_{pq} \otimes I_n) + (I_n \otimes K_{qp} \otimes I_n)(K_{np} \otimes I_{nq}) \\
&\quad \cdot(\Omega_{XY} \otimes \Omega_{YX}K_{pn} + \mu\nu' \otimes \Omega_{YX}K_{pn} + \Omega_{XY} \otimes \nu\tilde{\mu}') \\
&\quad \cdot K_{nq,np}(I_n \otimes K_{pq} \otimes I_n) \\
&= (I_n \otimes K_{qp} \otimes I_n)(\tilde{\Omega}_{XX} \otimes \Omega_{YY} + \tilde{\Omega}_{XX} \otimes \nu\nu' + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{YY}) \\
&\quad \cdot(I_n \otimes K_{pq} \otimes I_n) + (I_n \otimes K_{qp} \otimes I_n) \\
&\quad \cdot(K_{np}\Omega_{XY} \otimes \Omega_{YX}K_{pn} + \tilde{\mu}\nu' \otimes \Omega_{YX}K_{pn} + K_{np}\Omega_{XY} \otimes \nu\tilde{\mu}') \\
&\quad \cdot K_{nq,np}(I_n \otimes K_{pq} \otimes I_n) \\
&= (I_n \otimes K_{qp} \otimes I_n)(\tilde{\Omega}_{XX} \otimes \Omega_{YY} + \tilde{\Omega}_{XX} \otimes \nu\nu' + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{YY}) \\
&\quad \cdot(I_n \otimes K_{pq} \otimes I_n) + (I_n \otimes K_{qp} \otimes I_n) \\
&\quad \cdot K_{np,nq}(\Omega_{YX}K_{pn} \otimes K_{np}\Omega_{XY} + \Omega_{YX}K_{pn} \otimes \tilde{\mu}\nu' + \nu\tilde{\mu}' \otimes K_{np}\Omega_{XY}) \\
&\quad \cdot(I_n \otimes K_{pq} \otimes I_n).
\end{aligned}$$

The second term can then be simplified by substituting

$$(I_n \otimes K_{pq} \otimes I_n)(K_{nq} \otimes K_{pn})(I_q \otimes K_{nn} \otimes I_p) \quad \text{for } K_{np,nq}. \qquad \square$$

We shall now specialize to

$$\Omega = \Phi \otimes I_n,$$

with

$$\Phi = \begin{bmatrix} \Phi_{XX} & \Phi_{XY} \\ \Phi_{YX} & \Phi_{YY} \end{bmatrix}.$$

This yields:

$$\begin{aligned}
\mathscr{D}\{\text{vec}(X \otimes Y)\} = {} & \Phi_{XX} \otimes \Phi_{YY} \otimes I_{n^2} + \Phi_{XX} \otimes (K_{qn} \otimes I_n)(I_n \otimes \nu\nu') \\
& \cdot(K_{nq} \otimes I_n) + (I_p \otimes K_{qn})(\mu\mu' \otimes \Phi_{YY})(I_p \otimes K_{nq}) \otimes I_n \\
& + K_{pq}(\Phi_{YX} \otimes \Phi_{XY}) \otimes K_{nn} + (K_{pq} \otimes K_{nn}) \\
& \cdot\{\Phi_{YX} \otimes (K_{pn} \otimes I_n)(I_n \otimes \mu\nu')(K_{nq} \otimes I_n)\} \\
& + (K_{pq} \otimes K_{nn})\{(I_q \otimes K_{pn}) \\
& \cdot(\nu\mu' \otimes \Phi_{XY})(I_p \otimes K_{nq}) \otimes I_n\}.
\end{aligned} \tag{5.3}$$

Proof. In (5.1) we replace Ω_{XX}, Ω_{YY}, Ω_{XY} and Ω_{YX} by $\Phi_{XX} \otimes I_n$, $\Phi_{YY} \otimes I_n$, $\Phi_{XY} \otimes I_n$ and $\Phi_{YX} \otimes I_n$, respectively.

This leads to the following two terms:

$$\begin{aligned}
&(I_p \otimes K_{qn} \otimes I_n)(\Phi_{XX} \otimes I_n \otimes \Phi_{YY} \otimes I_n + \Phi_{XX} \otimes I_n \otimes \nu\nu' \\
&\quad + \mu\mu' \otimes \Phi_{YY} \otimes I_n)(I_p \otimes K_{nq} \otimes I_n) \\
&\qquad = \Phi_{XX} \otimes \Phi_{YY} \otimes I_{n^2} + \Phi_{XX} \otimes (K_{qn} \otimes I_n)(I_n \otimes \nu\nu')(K_{nq} \otimes I_n) \\
&\qquad\quad + (I_p \otimes K_{qn}) \otimes (\mu\mu' \otimes \Phi_{YY})(I_p \otimes K_{nq}) \otimes I_n
\end{aligned}$$

and

$$
\begin{aligned}
&(K_{pq} \otimes K_{nn})(I_q \otimes K_{pn} \otimes I_n)(\Phi_{YX} \otimes I_n \otimes \Phi_{XY} \otimes I_n \\
&\quad +\Phi_{YX} \otimes I_n \otimes \mu\nu' + \nu\mu' \otimes \Phi_{XY} \otimes I_n)(I_p \otimes K_{nq} \otimes I_n) \\
&\qquad = (K_{pq} \otimes K_{nn})(\Phi_{YX} \otimes \Phi_{XY} \otimes I_{n^2}) \\
&\qquad\quad +(K_{pq} \otimes K_{nn})\{\Phi_{YX} \otimes (K_{pn} \otimes I_n)(I_n \otimes \mu\nu')(K_{nq} \otimes I_n)\} \\
&\qquad\quad +(K_{pq} \otimes K_{nn})\{(I_q \otimes K_{pn})(\nu\mu' \otimes \Phi_{XY})(I_p \otimes K_{nq}) \otimes I_n\} \\
&\qquad = K_{pq}(\Phi_{YX} \otimes \Phi_{XY}) \otimes K_{nn} \\
&\qquad\quad +(K_{pq} \otimes K_{nn})\{\Phi_{YX} \otimes (K_{pn} \otimes I_n)(I_n \otimes \mu\nu')(K_{nq} \otimes I_n)\} \\
&\qquad\quad +(K_{pq} \otimes K_{nn})\{(I_q \otimes K_{pn})(\nu\mu' \otimes \Phi_{XY})(I_p \otimes K_{nq}) \otimes I_n\}. \qquad \square
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes Y)\} \\
&\quad = I_n \otimes \Phi_{YY} \otimes \Phi_{XX} \otimes I_n + I_n \otimes (K_{qp} \otimes I_n)(\Phi_{XX} \otimes \nu\nu')(K_{pq} \otimes I_n) \\
&\qquad +(I_n \otimes K_{qp})(\tilde{\mu}\tilde{\mu}' \otimes \Phi_{YY})(I_n \otimes K_{pq}) \otimes I_n \\
&\qquad +(K_{nq} \otimes K_{pn})(I_q \otimes K_{nn} \otimes I_p)\{(K_{qn} \otimes K_{np})(I_n \otimes \Phi_{YX} \otimes \Phi_{XY} \otimes I_n) \\
&\qquad +(\Phi_{YX} \otimes I_n)K_{pn} \otimes \tilde{\mu}\nu' + \nu\tilde{\mu}' \otimes K_{np}(\Phi_{XY} \otimes I_n)\} \\
&\qquad \cdot(I_n \otimes K_{pq} \otimes I_n).
\end{aligned} \tag{5.4}
$$

Proof. Replacing $\tilde{\Omega}_{XX}$ by $K_{np}(\Phi_{XX} \otimes I_n)K_{pn} = I_n \otimes \Phi_{XX}, \Omega_{YY}, \Omega_{XY}$ and Φ_{YX} by $\Phi_{YY} \otimes I_n,\ \Phi_{XY} \otimes I_n$ and $\Phi_{YX} \otimes I_n$, respectively, in (5.2) we get the two terms:

$$
\begin{aligned}
&(I_n \otimes K_{qp} \otimes I_p)(I_n \otimes \Phi_{XX} \otimes \Phi_{YY} \otimes I_n + I_n \otimes \Phi_{XX} \otimes \nu\nu' \\
&\quad +\tilde{\mu}\tilde{\mu}' \otimes \Phi_{YY} \otimes I_n)(I_n \otimes K_{pq} \otimes I_n) \\
&\qquad = I_n \otimes \Phi_{YY} \otimes \Phi_{XX} \otimes I_n + I_n \otimes (K_{qp} \otimes I_n)(\Phi_{XX} \otimes \nu\nu')(K_{pq} \otimes I_n) \\
&\qquad\quad + (I_n \otimes K_{qp})(\tilde{\mu}\tilde{\mu}' \otimes \Phi_{YY})(I_n \otimes K_{pq}) \otimes I_n,
\end{aligned}
$$

and

$$
\begin{aligned}
&(K_{nq} \otimes K_{pn})(I_q \otimes K_{nn} \otimes I_p)\{(\Phi_{YX} \otimes I_n)K_{pn} \otimes K_{np}(\Phi_{XY} \otimes I_n) \\
&\quad +(\Phi_{YX} \otimes I_n)K_{pn} \otimes \tilde{\mu}\nu' + \nu\tilde{\mu}' \otimes K_{np}(\Phi_{XY} \otimes I_n)\}(I_n \otimes K_{pq} \otimes I_n) \\
&\qquad = (K_{nq} \otimes K_{pn})(I_q \otimes K_{nn} \otimes I_p)\{(K_{qn} \otimes K_{np})(I_n \otimes \Phi_{YX} \otimes \Phi_{XY} \otimes I_n) \\
&\qquad\quad +(\Phi_{YX} \otimes I_n)K_{pn} \otimes \tilde{\mu}\nu' + \nu\tilde{\mu}' \otimes K_{np}(\Phi_{XY} \otimes I_n)\} \\
&\qquad\quad \cdot(I_n \otimes K_{pq} \otimes I_n). \qquad \square
\end{aligned}
$$

5.2.

From the results of the preceding section, under the assumptions made, we get for the general case where $E(\mathrm{vec}\, X) = \mu$ and $\mathscr{D}(\mathrm{vec}\, X) = \Omega_{XX}$:

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X \otimes X)\} \\
&\quad = (I_{n^2p^2} + K_{pp} \otimes K_{nn})(I_p \otimes K_{pn} \otimes I_n) \\
&\qquad \cdot(\Omega_{XX} \otimes \Omega_{XX} + \Omega_{XX} \otimes \mu\mu' + \mu\mu' \otimes \Omega_{XX})(I_p \otimes K_{np} \otimes I_n), \qquad (5.5)
\end{aligned}$$

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes X)\} \\
&\quad = (I_n \otimes K_{pp} \otimes I_n)(\tilde{\Omega}_{XX} \otimes \Omega_{XX} + \tilde{\Omega}_{XX} \otimes \mu\mu' + \tilde{\mu}\tilde{\mu}' \otimes \Omega_{XX}) \\
&\qquad \cdot(I_n \otimes K_{pp} \otimes I_n) + (K_{np} \otimes K_{pn})(I_p \otimes K_{nn} \otimes I_p) \\
&\qquad \cdot(\Omega_{XX}K_{pn} \otimes K_{np}\Omega_{XX} + \Omega_{XX}K_{pn} \otimes \tilde{\mu}\mu' + \mu\tilde{\mu}' \otimes K_{np}\Omega_{XX}) \\
&\qquad \cdot(I_n \otimes K_{pp} \otimes I_n). \qquad (5.6)
\end{aligned}$$

For the special case: $\Omega_{XX} = \Phi_{XX} \otimes I_n$, we get

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X \otimes X)\} \\
&\quad = \Phi_{XX} \otimes \Phi_{XX} \otimes I_{n^2} + \Phi_{XX} \otimes (K_{pn} \otimes I_n)(I_n \otimes \mu\mu')(K_{np} \otimes I_n) \\
&\qquad +(I_p \otimes K_{pn})(\mu\mu' \otimes \Phi_{XX})(I_p \otimes K_{np}) \otimes I_n \\
&\qquad +K_{pp}(\Phi_{XX} \otimes \Phi_{XX}) \otimes K_{nn} + (K_{pp} \otimes K_{nn}) \\
&\qquad \cdot\{\Phi_{XX} \otimes (K_{pn} \otimes I_n)(I_n \otimes \mu\mu')(K_{np} \otimes I_n)\} \\
&\qquad +(K_{pp} \otimes K_{nn})\{(I_p \otimes K_{pn})(\mu\mu' \otimes \Phi_{XX})(I_p \otimes K_{np}) \otimes I_n\} \qquad (5.7)
\end{aligned}$$

by making the appropriate substitutions in (5.3).

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes X)\} \\
&\quad = I_n \otimes \Phi_{XX} \otimes \Phi_{XX} \otimes I_n + I_n \otimes (K_{pp} \otimes I_n)(\Phi_{XX} \otimes \mu\mu')(K_{pp} \otimes I_n) \\
&\qquad +(I_n \otimes K_{pp})(\tilde{\mu}\tilde{\mu}' \otimes \Phi_{XX})(I_n \otimes K_{pp}) \otimes I_n \\
&\qquad +(K_{np} \otimes K_{pn})(I_p \otimes K_{nn} \otimes I_p)\{(K_{pn} \otimes K_{np})(I_n \otimes \Phi_{XX} \otimes \Phi_{XX} \otimes I_n) \\
&\qquad +(\Phi_{XX} \otimes I_n)K_{pn} \otimes \tilde{\mu}\mu' + \mu\tilde{\mu}' \otimes K_{np}(\Phi_{XX} \otimes I_n)\} \\
&\qquad \cdot(I_n \otimes K_{pp} \otimes I_n) \qquad (5.8)
\end{aligned}$$

by making the appropriate substitutions in (5.4).

For $\Omega_{XX} = V \otimes U$, hence $\tilde{\Omega}_{XX} = U \otimes V$, the results (5.7) and (5.8) change to

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X \otimes X)\} \\
&\quad = (I_{n^2p^2} + K_{pp} \otimes K_{nn})\{V \otimes V \otimes U \otimes U + V \otimes (K_{pn} \otimes I_n)(U \otimes \mu\mu') \\
&\qquad \cdot(K_{np} \otimes I_n) + (I_p \otimes K_{pn})(\mu\mu' \otimes V)(I_p \otimes K_{np}) \otimes U\} \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes X)\} \\
&\quad = U \otimes V \otimes V \otimes U + U \otimes (K_{pp} \otimes I_n)(V \otimes \mu\mu')(K_{pp} \otimes I_n) \\
&\qquad +(I_n \otimes K_{pp})(\tilde{\mu}\tilde{\mu}' \otimes V)(I_n \otimes K_{pp}) \otimes U
\end{aligned}$$

$$+(K_{np} \otimes K_{pn})(I_p \otimes K_{nn} \otimes I_p)\{(K_{pn} \otimes K_{np})(U \otimes V \otimes V \otimes U)$$
$$+(V \otimes U)K_{pn} \otimes \tilde{\mu}\mu' + \mu\tilde{\mu}' \otimes K_{np}(V \otimes U)\}(I_n \otimes K_{pp} \otimes I_n). \quad (5.10)$$

For the general specification, $\Omega_{XX} = \sum_{ij}(E_{ij} \otimes \Psi_{ij})$, hence

$$\tilde{\Omega}_{XX} = \sum_{ij}(\Psi_{ij} \otimes E_{ij}).$$

Eqs. (5.5) and (5.6) yield

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X \otimes X)\} \\
&\quad = (I_{n^2p^2} + K_{pp} \otimes K_{nn}) \sum_{ijk\ell}(E_{ij} \otimes E_{k\ell} \otimes \Psi_{ij} \otimes \Psi_{k\ell}) \\
&\qquad +(I_{n^2p^2} + K_{pp} \otimes K_{nn})(I_p \otimes K_{pn} \otimes I_n) \\
&\qquad \cdot \sum_{ij}(E_{ij} \otimes \Psi_{ij} \otimes \mu\mu' + \mu\mu' \otimes E_{ij} \otimes \Psi_{ij})(I_p \otimes K_{np} \otimes I_n) \qquad (5.11)
\end{aligned}$$

$$\begin{aligned}
&\mathscr{D}\{\mathrm{vec}(X' \otimes X)\} \\
&\quad = \sum_{ijk\ell}(\Psi_{ij} \otimes E_{k\ell} \otimes E_{ij} \otimes \Psi_{k\ell}) \\
&\qquad +(I_n \otimes K_{pp} \otimes I_n) \sum_{ij}(\Psi_{ij} \otimes E_{ij} \otimes \mu\mu' + \tilde{\mu}\tilde{\mu}' \otimes E_{ij} \otimes \Psi_{ij}) \\
&\qquad \cdot(I_n \otimes K_{pp} \otimes I_n) + (K_{np} \otimes K_{pn})(I_p \otimes K_{nn} \otimes I_p)(K_{pn} \otimes K_{np}) \\
&\qquad \cdot \sum_{ijk\ell}(\Psi_{ij} \otimes E_{ij} \otimes E_{k\ell} \otimes \Psi_{k\ell})(I_n \otimes K_{pp} \otimes I_n) \\
&\qquad +(K_{np} \otimes K_{pn})(I_p \otimes K_{nn} \otimes I_p) \sum_{ij}\{(E_{ij} \otimes \Psi_{ij})K_{pn} \otimes \tilde{\mu}\mu' \\
&\qquad +\mu\tilde{\mu}' \otimes K_{np}(E_{ij} \otimes \Psi_{ij})\}(I_n \otimes K_{pp} \otimes I_n). \qquad (5.12)
\end{aligned}$$

6. Expected values of matrix bilinear forms in jointly distributed random matrices

Consider again $Z = (X, Y)$ with $(n \times p)X$ and $(n \times q)Y$. Let $E(Z) = (M, N)$ and

$$\mathscr{D}(\mathrm{vec}\, Z) = \Omega = \begin{bmatrix} \Omega_{XX} & \Omega_{XY} \\ \Omega_{YX} & \Omega_{YY} \end{bmatrix}.$$

Let further $\Omega = \Phi \otimes I_n$, with

$$\Phi = \begin{bmatrix} \Phi_{XX} & \Phi_{XY} \\ \Phi_{YX} & \Phi_{YY} \end{bmatrix}.$$

We consider the following three matrix bilinear forms: $X'AY$, XAY' and XAY. Realize that A is a fixed (generic) matrix. Then

$$E(X'AY) = (\operatorname{tr} A)\Omega_{XY} + M'AN.$$

Proof.

$$\begin{aligned}
\operatorname{vec} E(X'AY) &= E(Y' \otimes X')\operatorname{vec} A \\
&= \left\{\sum_{ij}(E'_{ij} \otimes I_n)\Omega_{XY}(I_q \otimes E'_{ij})\right\}' \operatorname{vec} A + (N \otimes M)'\operatorname{vec} A \\
&= \sum_{ij}(I_q \otimes E_{ij})\Omega_{YX}(E_{ij} \otimes I_n)\operatorname{vec} A + \operatorname{vec} M'AN \\
&= \sum_{ij}(\Phi_{YX}E_{ij} \otimes E_{ij})\operatorname{vec} A + \operatorname{vec} M'AN \\
&= \operatorname{vec} \sum_{ij} E_{ij}AE'_{ij}\Phi_{XY} + \operatorname{vec} M'AN \\
&= \operatorname{vec} \left(\sum_j a_{jj}\right) \sum_i E_{ii}\Phi_{XY} + \operatorname{vec} M'AN \\
&= \operatorname{vec}(\operatorname{tr} A)\Phi_{XY} + \operatorname{vec} M'AN.
\end{aligned}$$

Hence

$$E(X'AY) = (\operatorname{tr} A)\Phi_{XY} + M'AN\cdot$$

Here E_{ij} is a $(p \times n)$ basis matrix $(i = 1, \ldots, p;\ j = 1, \ldots, n)$. We used (4.1). $\square$

$$E(XAY') = (\operatorname{tr} A\Phi_{YX})I_n + MAN'.$$

Proof.

$$\begin{aligned}
\operatorname{vec} E(XAY') &= E(Y \otimes X)\operatorname{vec} A \\
&= \sum_{ij}(E'_{ij} \otimes I_n)\Omega_{XY}(I_q \otimes E'_{ij})\operatorname{vec} A + (N \otimes M)\operatorname{vec} A \\
&= \sum_{ij}(E'_{ij}\Phi_{XY} \otimes E'_{ij})\operatorname{vec} A + (N \otimes M)\operatorname{vec} A \\
&= \operatorname{vec} \sum_{ij} E'_{ij}A\Phi_{YX}E_{ij} + \operatorname{vec} MAN' \\
&= \operatorname{vec} \sum_i (A\Phi_{YX})_{ii} \sum_j E_{jj} + \operatorname{vec} MAN'
\end{aligned}$$

$$= (\mathrm{tr}\, A\Phi_{YX})\mathrm{vec}\, I_n + \mathrm{vec}\, MAN'$$

Here E_{ij} is a $(p \times n)$ basis matrix $(i = 1, \ldots, p;\ \ j = 1, \ldots, n)$. Again we used (4.1). □

$$E(XAY) = A'\Phi_{XY} + MAN.$$

Proof.

$$\begin{aligned}
\mathrm{vec}\, E(XAY) &= E(Y' \otimes X)\mathrm{vec}\, A \\
&= \sum_{ij} (E'_{ij} \otimes I_n)\Omega_{XY}(E'_{ij} \otimes I_n)K_{pn}\mathrm{vec}\, A + (N' \otimes M)\mathrm{vec}\, A \\
&= \mathrm{vec} \sum_{ij} A'E_{ij}\Phi_{YX}E_{ij} + \mathrm{vec}\, MAN \\
&= \mathrm{vec}\, A' \sum_{ij} (\Phi_{XY})_{ij}E_{ij} + \mathrm{vec}\, MAN \\
&= \mathrm{vec}(A'\Phi_{XY} + MAN).
\end{aligned}$$

Here E_{ij} is a $(p \times q)$ basis matrix $(i = 1, \ldots, p;\ \ j = 1, \ldots, q)$. We used (4.2). □

We examined the three major forms $X'AY$, XAY' and XAY. From these can be obtained the forms: $Y'AX$, YAX', YAX and $Y'AX'$, by substitution or transposition.
This yields:

$$\begin{aligned}
&E(Y'AX) = (\mathrm{tr}\, A)\Phi_{YX} + N'AM, \\
&E(YAX') = (\mathrm{tr}\, A\Phi_{XY})I_n + NAM', \\
&E(YAX) = A'\Phi_{YX} + NAM, \\
&E(Y'AX') = \Phi_{YX}A' + N'AM'.
\end{aligned}$$

Special cases are:

$$E(X'AX) = (\mathrm{tr}\, A)\Phi_{XX} + M'AM, \tag{6.1}$$

$$E(XAX') = (\mathrm{tr}\, A\Phi_{XX})I_n + MAM', \tag{6.2}$$

$$E(XAX) = A'\Phi_{XX} + MAM. \tag{6.3}$$

Slightly less special variants of (6.1)–(6.3) for $\mathscr{D}(\mathrm{vec}\, X) = V \otimes U$ with psd $(n \times n)U$ and $(p \times p)V$ are:

$$E(X'AX) = (\mathrm{tr}\, AU)V + M'AM, \tag{6.4}$$

$$E(XAX') = (\mathrm{tr}\, AV)U + MAM', \tag{6.5}$$

$$E(XAX) = UA'V + MAM. \tag{6.6}$$

They follow directly from (4.5) and (4.6) by (de-)vectorization.

In fact (6.1)–(6.3) also follow from (6.4)–(6.6).

For the general decomposition $\Omega_{XX} = \sum_{ij}(E_{ij} \otimes \Psi_{ij})$, with E_{ij} a $(p \times p)$ basis matrix, we obtain

$$E(X'AX) = \sum_{ij}(\mathrm{tr}\, A'\Psi_{ij})E_{ij} + M'AM, \tag{6.7}$$

$$E(XAX') = \sum_{ij} a_{ij}\Psi_{ij} + MAM', \tag{6.8}$$

$$E(XAX) = \sum_{ij} \Psi_{ij}A'E_{ij} + MAM. \tag{6.9}$$

They follow from (4.11) and (4.12) by (de-)vectorization.

Notice that neither E_{ij} nor Ψ_{ij} are symmetric unless $i = j$.

Remarks.

1. It will be clear that the results of this section and Section 4 are closely related. We elected to base this section on the fourth. But the reverse order is also attractive. One can, in fact, do both simultaneously.
2. An interesting application concerns $S \sim W_p(\Sigma, n)$. Then

$$E(SBS) = n\Sigma B'\Sigma + n(\mathrm{tr}\, B\Sigma)\Sigma + n^2\Sigma B\Sigma.$$

The property is proved by means of (6.7), taking $X' = X = \Sigma^{-1/2}S\Sigma^{-1/2}$ and $A = \Sigma^{1/2}B\Sigma^{1/2}$. One could as well use (6.8) or (6.9).

Now $\mathscr{D}(\mathrm{vec}\, X) = n\sum_{ij}\{E_{ij} \otimes (\delta_{ij}I_p + E'_{ij})\}$, δ_{ij} being the Kronecker delta and $E(X) = nI_p$. See also [12, Corollary B5] for a different derivation, with $\Sigma = I_p$. Shalabh [27, Appendix] mistakenly gives

$$E(SBS) = n(n+1)\Sigma B\Sigma + n(\mathrm{tr}\, B\Sigma)\Sigma.$$

He quotes earlier results of Srivastava and Tiwari [28].

7. The covariance of two matrix quadratic forms in a normally distributed random matrix

Consider $\mathrm{vec}\, X \sim N_{np}(\mathrm{vec}\, M,\ \ V \otimes U)$, and define $S_A := X'AX$ and $S_B := X'BX$. We shall then find for the covariance of S_A and S_B:

$$C(\mathrm{vec}\, S_A, \mathrm{vec}\, S_B)$$

$$
\begin{aligned}
&= (\operatorname{tr} A'UBU)V \otimes V + M'A'UBM \otimes V + V \otimes M'AUB'M \\
&\quad + K_{pp}\{(\operatorname{tr} AUBU)V \otimes V + V \otimes M'A'UB'M \\
&\qquad + M'AUBM \otimes V\}.
\end{aligned} \tag{7.1}
$$

Proof. Clearly,

$$\operatorname{vec} X' \sim N_{pn}(\operatorname{vec} M', U \otimes V).$$

Further,

$$
\begin{aligned}
\operatorname{vec} S_A &= \operatorname{vec} X'AX \\
&= \operatorname{vec}\{(X' \otimes X')\operatorname{vec} A\} \\
&= (\operatorname{vec} A \otimes I_{p^2})' \operatorname{vec}(X' \otimes X').
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&C(\operatorname{vec} S_A, \operatorname{vec} S_B) \\
&\quad = (\operatorname{vec} A \otimes I_{p^2})' \mathscr{D}\{\operatorname{vec}(X' \otimes X')\}(\operatorname{vec} B \otimes I_{p^2}) \\
&\quad = (\operatorname{vec} A \otimes I_{p^2})'[(I_{p^2n^2} + K_{nn} \otimes K_{pp})(I_n \otimes K_{np} \otimes I_p)(U \otimes V \otimes U \otimes V \\
&\qquad + U \otimes V \otimes \tilde{\mu}\tilde{\mu}' + \tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)](\operatorname{vec} B \otimes I_{p^2}),
\end{aligned}
$$

where $\tilde{\mu} := \operatorname{vec} M'$. We used (5.5).

Consider first

$$
\begin{aligned}
&(\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(U \otimes V \otimes U \otimes V + U \otimes V \otimes \tilde{\mu}\tilde{\mu}' \\
&\quad + \tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2}) \\
&\qquad = (\operatorname{vec} A \otimes I_{p^2})'(U \otimes U \otimes V \otimes V)(\operatorname{vec} B \otimes I_{p^2}) \\
&\qquad\quad + (\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(U \otimes V \otimes \tilde{\mu}\tilde{\mu}') \\
&\qquad\quad \cdot (I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2}) \\
&\qquad\quad + (\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(\tilde{\mu}\tilde{\mu}' \otimes U \otimes V) \\
&\qquad\quad \cdot (I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2}) \\
&\qquad = (\operatorname{tr} A'UBU)V \otimes V \\
&\qquad\quad + (\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)K_{np,np}(\tilde{\mu}\tilde{\mu}' \otimes U \otimes V) \\
&\qquad\quad \cdot K_{np,np}(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2}) \\
&\qquad\quad + (\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(\tilde{\mu}\tilde{\mu}' \otimes U \otimes V) \\
&\qquad\quad \cdot (I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2}) \\
&\qquad = (\operatorname{tr} A'UBU)V \otimes V \\
&\qquad\quad + (\operatorname{vec} A \otimes I_{p^2})'(K_{nn} \otimes K_{pp})(I_n \otimes K_{np} \otimes I_p) \\
&\qquad\quad \cdot (\tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(K_{nn} \otimes K_{pp})
\end{aligned}
$$

$$
\begin{aligned}
&\cdot(\operatorname{vec} B \otimes I_{p^2}) + (\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)\\
&\cdot(\tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2})\\
={}& (\operatorname{tr} A'UBU)V \otimes V + (\operatorname{vec} A' \otimes K_{pp})'(I_n \otimes K_{np} \otimes I_p)\\
&\cdot(\tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)\\
&\cdot(\operatorname{vec} B' \otimes K_{pp}) + (\operatorname{vec} A \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)\\
&\cdot(\tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2})\\
={}& (\operatorname{tr} A'UBU)V \otimes V + (\operatorname{vec} A \otimes I_p)'(I_n \otimes K_{np})\\
&\cdot(\tilde{\mu}\tilde{\mu}' \otimes U)(I_n \otimes K_{pn})(\operatorname{vec} B \otimes I_p) \otimes V\\
&+K_{pp}[(\operatorname{vec} A' \otimes I_p)'(I_n \otimes K_{np})(\tilde{\mu}\tilde{\mu}' \otimes U)\\
&\cdot(I_n \otimes K_{pn})(\operatorname{vec} B' \otimes I_p) \otimes V]K_{pp}\\
={}& (\operatorname{tr} A'UBU)V \otimes V + M'A'UBM \otimes V\\
&+K_{pp}(M'AUB'M \otimes V)K_{pp}\\
={}& (\operatorname{tr} A'UBU)V \otimes V + M'A'UBM \otimes V + V \otimes M'AUB'M.
\end{aligned}
$$

We used (2.14).

Consider next

$$
\begin{aligned}
&(\operatorname{vec} A \otimes I_{p^2})'(K_{nn} \otimes K_{pp})(I_n \otimes K_{np} \otimes I_p)(U \otimes V \otimes U \otimes V\\
&\quad +U \otimes V \otimes \tilde{\mu}\tilde{\mu}' + \tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2})\\
&\quad = (\operatorname{vec} A' \otimes K_{pp})'(I_n \otimes K_{np} \otimes I_p)(U \otimes V \otimes U \otimes V + U \otimes V \otimes \tilde{\mu}\tilde{\mu}'\\
&\qquad +\tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2})\\
&\quad = K_{pp}(\operatorname{vec} A' \otimes I_{p^2})'(I_n \otimes K_{np} \otimes I_p)(U \otimes V \otimes U \otimes V\\
&\qquad +U \otimes V \otimes \tilde{\mu}\tilde{\mu}' + \tilde{\mu}\tilde{\mu}' \otimes U \otimes V)(I_n \otimes K_{pn} \otimes I_p)(\operatorname{vec} B \otimes I_{p^2})\\
&\quad = (\operatorname{tr} AUBU)K_{pp}(V \otimes V)\\
&\qquad +K_{pp}(V \otimes M'A'UB'M) + K_{pp}(M'AUBM \otimes V).
\end{aligned}
$$

We used (2.14), (2.11) and (2.5). □

Corollary 1.

$$
\begin{aligned}
\mathscr{D}(\operatorname{vec} S_A) = {}& (\operatorname{tr} A'UAU)V \otimes V + M'A'UAM \otimes V + V \otimes M'AUA'M\\
&+K_{pp}\{(\operatorname{tr} AUAU)V \otimes V + V \otimes M'A'UA'M\\
&+M'AUAM \otimes V\}
\end{aligned}
$$

for $S_A := X'AX$.

See [26] for a different proof.

Corollary 2.

$$\begin{aligned}&C(\text{vec}\, S_A,\ \text{vec}\, S_B)\\&\quad=(\text{tr}\, A'B)\Phi_{XX}\otimes\Phi_{XX}+M'A'BM\otimes\Phi_{XX}+\Phi_{XX}\otimes M'AB'M\\&\qquad+K_{pp}\{(\text{tr}\, AB)\Phi_{XX}\otimes\Phi_{XX}+\Phi_{XX}\otimes M'A'B'M+M'ABM\otimes\Phi_{XX}\},\end{aligned}$$

when $\text{vec}\, X\sim N_{np}(\text{vec}\, M,\ \Phi_{XX}\otimes I_n)$.

Corollary 3.

$$\begin{aligned}\mathscr{D}(\text{vec}\, S_A)=&(\text{tr}\, A'A)\Phi_{XX}\otimes\Phi_{XX}+M'A'AM\otimes\Phi_{XX}+\Phi_{XX}\otimes M'AA'M\\&+K_{pp}\{(\text{tr}\, A^2)\Phi_{XX}\otimes\Phi_{XX}+\Phi_{XX}\otimes(M'A^2M)'\\&+M'A^2M\otimes\Phi_{XX}\}\end{aligned}$$

when $\text{vec}\, X\sim N_{np}(\text{vec}\, M,\ \Phi_{XX}\otimes I_n)$.

Remark 1. Neudecker and Wansbeek [26] derived (7.1) from $E(X'AXCX'BX)$, which they established first. In the present paper, this expected value will come later, viz. in Section 9.

Remark 2. It is not difficult to extend the result of this section to the general variance expression $\mathscr{D}(\text{vec}\, X)=\sum_{ij}(E_{ij}\otimes\Psi_{ij})\ (i,j=1,\dots,p)$. This will not be pursued here.

8. Fourth-order moments for a normally distributed random matrix

Consider an $(n\times p)$ normally distributed matrix X:

$$\text{vec}\, X\sim N_{np}(\text{vec}\, M,\ V\otimes U).$$

Then:

$$\begin{aligned}E(X\otimes X\otimes X\otimes X)=&(uv'+M\otimes M)\otimes(uv'+M\otimes M)\\&+C_2^n[(uv'+M\otimes M)\otimes(uv'+M\otimes M)]C_2^p\\&+C_3^nC_2^n[(uv'+M\otimes M)\otimes(uv'+M\otimes M)]C_2^pC_3^p\\&-2M\otimes M\otimes M\otimes M,\end{aligned}\tag{8.1}$$

where $u:=\text{vec}\, U,\ v:=\text{vec}\, V$,

$$C_2^i:=I_i\otimes K_{ii}\otimes I_i,\quad C_3^i:=I_i^2\otimes K_{ii}\quad(i=n,p).$$

Proof. For $y=\text{vec}\, Y\sim N_{np}(0,\ V\otimes U)$, we get

$$
\begin{aligned}
&E \operatorname{vec}(Y \otimes Y \otimes Y \otimes Y) \\
&\quad = (I_{p^2} \otimes K_{p^2,n^2} \otimes I_n^2) E[\operatorname{vec}(Y \otimes Y) \otimes \operatorname{vec}(Y \otimes Y)] \\
&\quad = (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2}) \\
&\qquad \cdot (I_p \otimes K_{pn} \otimes I_n \otimes I_p \otimes K_{pn} \otimes I_n) E(y \otimes y \otimes y \otimes y) \\
&\quad = (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2})(I_p \otimes K_{pn} \otimes I_n \otimes I_p \otimes K_{pn} \otimes I_n) \\
&\qquad \cdot \operatorname{vec}[(I_{n^2p^2} + K_{np,np}) \cdot (V \otimes U \otimes V \otimes U) \\
&\qquad + \operatorname{vec}(V \otimes U) \otimes \operatorname{vec}(V \otimes U)] \\
&\quad = (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2}) \operatorname{vec}\{(I_p \otimes K_{pn} \otimes I_n) \\
&\qquad \cdot (I_{n^2p^2} + K_{np,np})(V \otimes U \otimes V \otimes U) \\
&\qquad \cdot (I_p \otimes K_{np} \otimes I_n)\} + (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2}) \\
&\qquad \cdot (\operatorname{vec} V \otimes \operatorname{vec} U \otimes \operatorname{vec} V \otimes \operatorname{vec} U) \\
&\quad = (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2}) \operatorname{vec}(V \otimes V \otimes U \otimes U) + v \otimes v \otimes u \otimes u \\
&\qquad + (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2}) \operatorname{vec}(K_{pp} \otimes K_{nn}) \\
&\qquad \cdot (I_p \otimes K_{pn} \otimes I_n)(V \otimes U \otimes V \otimes U)(I_p \otimes K_{np} \otimes I_n) \\
&\quad = \operatorname{vec}(V \otimes V) \otimes \operatorname{vec}(U \otimes U) + \operatorname{vec} vv' \otimes \operatorname{vec} uu' \\
&\qquad + (I_{p^2} \otimes K_{p^2,n^2} \otimes I_{n^2}) \operatorname{vec}(K_{pp} \otimes K_{nn})(V \otimes V \otimes U \otimes U) \\
&\quad = \operatorname{vec}\{\operatorname{vec}(U \otimes U)\}\{\operatorname{vec}(V \otimes V)\}' + \operatorname{vec}\{(\operatorname{vec} uu')(\operatorname{vec} vv')'\} \\
&\qquad + \operatorname{vec} K_{pp}(V \otimes V) \otimes \operatorname{vec} K_{nn}(U \otimes U) \\
&\quad = \operatorname{vec}\{C_2^n(u \otimes u)\}\{C_2^p(v \otimes v)\}' + \operatorname{vec}\{\operatorname{vec} K_{nn}(U \otimes U)\} \\
&\qquad \cdot \{\operatorname{vec} K_{pp}(V \otimes V)\}' + \operatorname{vec}(u \otimes u)(v \otimes v)' \\
&\quad = \operatorname{vec} C_2^n(uv' \otimes uv')C_2^p + \operatorname{vec}\{(I_{n^2} \otimes K_{nn})\operatorname{vec}(U \otimes U)\} \\
&\qquad \cdot (I_{p^2} \otimes K_{pp})\operatorname{vec}(V \otimes V)\}' + \operatorname{vec}(uv' \otimes uv') \\
&\quad = \operatorname{vec} C_2^n(uv' \otimes uv')C_2^p + \operatorname{vec} C_3^n C_2^n(uv' \otimes uv')C_2^p C_3^p + \operatorname{vec}(uv' \otimes uv').
\end{aligned}
$$

Hence

$$
\begin{aligned}
E(Y \otimes Y \otimes Y \otimes Y) = {} & uv' \otimes uv' + C_2^n(uv' \otimes uv')C_2^p \\
& + C_3^n C_2^n(uv' \otimes uv')C_2^p C_3^p.
\end{aligned}
$$

Use was made of (2.8) and (3.5). Take now $X = Y + M$. Then

$$
\begin{aligned}
&E(X \otimes X \otimes X \otimes X) \\
&\quad = E(Y \otimes Y \otimes Y \otimes Y) + E(Y \otimes Y \otimes M \otimes M) \\
&\qquad + E(Y \otimes M \otimes Y \otimes M) + E(Y \otimes M \otimes M \otimes Y) \\
&\qquad + E(M \otimes Y \otimes Y \otimes M) + E(M \otimes Y \otimes M \otimes Y) \\
&\qquad + E(M \otimes M \otimes Y \otimes Y) + M \otimes M \otimes M \otimes M
\end{aligned}
$$

$$
\begin{aligned}
&= uv' \otimes uv' + C_2^n(uv' \otimes uv')C_2^p + C_3^n C_2^n(uv' \otimes uv')C_2^p C_3^p \\
&\quad + uv' \otimes M \otimes M + C_2^n(uv' \otimes M \otimes M)C_2^p \\
&\quad + (I_n \otimes K_{n^2,n})(uv' \otimes M \otimes M)(I_p \otimes K_{p,p^2}) + M \otimes uv' \otimes M \\
&\quad + C_2^n(M \otimes M \otimes uv')C_2^p + M \otimes M \otimes uv' + M \otimes M \otimes M \otimes M \\
&= (uv' + M \otimes M) \otimes (uv' + M \otimes M) \\
&\quad + C_3^n C_2^n\{(uv' + M \otimes M) \otimes (uv' + M \otimes M)\}C_2^p C_3^p \\
&\quad - C_3^n C_2^n(M \otimes M \otimes M \otimes M)C_2^p C_3^p \\
&\quad + C_2^n\{(uv' + M \otimes M) \otimes (uv' + M \otimes M)\}C_2^p \\
&\quad - C_2^n(M \otimes M \otimes M \otimes M)C_2^p \\
&= (uv' + M \otimes M) \otimes (uv' + M \otimes M) \\
&\quad + C_2^n\{(uv' + M \otimes M) \otimes (uv' + M \otimes M)\}C_2^p \\
&\quad + C_3^n C_2^n\{(uv' + M \otimes M) \otimes (uv' + M \otimes M)\}C_2^p C_3^p \\
&\quad - 2M \otimes M \otimes M \otimes M.
\end{aligned}
$$

We used the properties $E(Y \otimes Y) = uv'$, $E(Y \otimes Y \otimes Y) = 0$, and (2.13). □

Note 1. For $\operatorname{vec} X \sim N_{np}(\operatorname{vec} M, \Phi_{XX} \otimes I_n)$ one replaces u by $\operatorname{vec} I_n$ and v by $\operatorname{vec} \Phi_{XX}$ in (8.1).

Note 2. For $\operatorname{vec} X \sim N_{np}(\operatorname{vec} M, \Omega)$ we get

$$
\begin{aligned}
&E(X \otimes X \otimes X \otimes X) \\
&= \left(\sum_{ij} \psi_{ij} e'_{ij} + M \otimes M\right) \otimes \left(\sum_{ij} \psi_{ij} e'_{ij} + M \otimes M\right) \\
&\quad + C_2^n \left\{\left(\sum_{ij} \psi_{ij} e'_{ij} + M \otimes M\right) \otimes \left(\sum_{ij} \psi_{ij} e'_{ij} + M \otimes M\right)\right\} C_2^p \\
&\quad + C_3^n C_2^n \left\{\left(\sum_{ij} \psi_{ij} e'_{ij} + M \otimes M\right) \otimes \left(\sum_{ij} \psi_{ij} e'_{ij} + M \otimes M\right)\right\} C_2^p C_3^p \\
&\quad - 2M \otimes M \otimes M \otimes M,
\end{aligned}
$$

where $\Omega = \sum_{ij}(E_{ij} \otimes \Psi_{ij})$, $\psi_{ij} := \operatorname{vec} \Psi_{ij}$ and $e_{ij} := \operatorname{vec} E_{ij}$.

Note 3. Result (8.1) has been established by Neudecker and Wansbeek [26] for the case $M = O$.

9. Expected values of matrix bilinear forms in matrix quadratic forms under normality

Consider the two $(p \times p)$ matrix quadratic forms $S_A := X'AX$ and $S_B := X'BX$, and the $(p \times p)$ matrix bilinear form $S_A D S_B'$.

The matrices A, B and D are fixed of orders $n \times n, n \times n$ and $p \times p$, respectively. The $(n \times p)$ random matrix X obeys a normal law: $\operatorname{vec} X \sim N(\operatorname{vec} M, V \otimes U)$, with $(p \times p)V$ and $(n \times n)U$, both positive semidefinite.

Our aim is to find the expected value $E(S_A D S_B')$. We get the following result:

$$
\begin{aligned}
&E(S_A D S_B') \\
&\quad = \{(\operatorname{tr} AU)V + M'AM\}D\{(\operatorname{tr} BU)V + M'B'M\} \\
&\qquad +(\operatorname{tr} AUBU)VD'V + VD'M'A'UB'M + M'AUBMD'V \\
&\qquad +(\operatorname{tr} AUB'U)(\operatorname{tr} DV)V + (\operatorname{tr} AMDM'B'U)V \\
&\qquad +(\operatorname{tr} DV)M'AUB'M. \qquad (9.1)
\end{aligned}
$$

Proof. By double vectorization we obtain

$$
\begin{aligned}
&\operatorname{vec} E(S_A D S_B') \\
&\quad = \operatorname{vec}\{E(S_B \otimes S_A)\operatorname{vec} D\} \\
&\quad = (\operatorname{vec} D \otimes I_{p^2})' E\operatorname{vec}(S_B \otimes S_A) \\
&\quad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p E(\operatorname{vec} S_B \otimes \operatorname{vec} S_A) \\
&\quad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p E(X \otimes X \otimes X \otimes X)'(\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\quad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p \{(vu' + M' \otimes M') \otimes (vu' + M' \otimes M')\} \\
&\qquad \cdot(\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\qquad +(\operatorname{vec} D \otimes I_{p^2})' C_2^p C_2^p \{(vu' + M' \otimes M') \otimes (vu' + M' \otimes M')\} \\
&\qquad \cdot C_2^n(\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\qquad +(\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \{(vu' + M' \otimes M') \otimes (vu' + M' \otimes M')\} \\
&\qquad \cdot C_2^n C_3^n(\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\qquad -2(\operatorname{vec} D \otimes I_{p^2})' C_2^p (M \otimes M \otimes M \otimes M)'(\operatorname{vec} B \otimes \operatorname{vec} \mathrm{A}),
\end{aligned}
$$

where $u := \operatorname{vec} U$, $v := \operatorname{vec} V$ and C_2^p, C_2^n, C_3^p and C_3^n were defined in (2.13). Further, we used (2.8) and (8.1).

We shall develop these four terms:

$$
\begin{aligned}
&(\operatorname{vec} D \otimes I_{p^2})' C_2^p \{(vu' + M' \otimes M') \otimes (vu' + M' \otimes M')\}(\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\quad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p \{(vu' + M' \otimes M')\operatorname{vec} B \otimes (vu' + M' \otimes M')\operatorname{vec} A\} \\
&\quad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p [\{(\operatorname{tr} BU)v + \operatorname{vec} M'BM\} \\
&\qquad \otimes\{(\operatorname{tr} AU)v + \operatorname{vec} M'AM\}]
\end{aligned}
$$

$$
\begin{aligned}
&= (\operatorname{vec} D \otimes I_{p^2})' C_2^p \{(\operatorname{tr} AU)(\operatorname{tr} BU)(v \otimes v) + (\operatorname{tr} AU)(\operatorname{vec} M'BM \otimes v) \\
&\quad + (\operatorname{tr} BU)(v \otimes \operatorname{vec} M'AM) + \operatorname{vec} M'BM \otimes \operatorname{vec} M'AM\} \\
&= (\operatorname{vec} D \otimes I_{p^2})' \{(\operatorname{tr} AU)(\operatorname{tr} BU)\operatorname{vec}(V \otimes V) + (\operatorname{tr} AU)\operatorname{vec}(M'BM \otimes V) \\
&\quad + (\operatorname{tr} BU)\operatorname{vec}(V \otimes M'AM) + \operatorname{vec}(M'BM \otimes M'AM)\} \\
&= (\operatorname{tr} AU)(\operatorname{tr} BU)\operatorname{vec} VDV + (\operatorname{tr} AU)\operatorname{vec} VDM'B'M \\
&\quad + (\operatorname{tr} BU)\operatorname{vec} M'AMDV + \operatorname{vec} M'AMDM'B'M \\
&= \operatorname{vec}\{(\operatorname{tr} AU)V + M'AM\} D \{(\operatorname{tr} BU)V + M'B'M\};
\end{aligned}
$$

$$
\begin{aligned}
&(\operatorname{vec} D \otimes I_{p^2})' C_2^p C_2^p \{(vu' + M' \otimes M') \\
&\quad \otimes (vu' + M' \otimes M')\} C_2^n (\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' \{(vu' + M' \otimes M') \otimes (vu' + M' \otimes M')\} \operatorname{vec}(B \otimes A) \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' \operatorname{vec}(vu' + M' \otimes M')(B \otimes A)(uv' + M \otimes M) \\
&\qquad = \operatorname{vec}\{(vu' + M' \otimes M')(B \otimes A)(uv' + M \otimes M) \operatorname{vec} D\} \\
&\qquad = (vu' + M' \otimes M')(B \otimes A)(uv' + M \otimes M) \operatorname{vec} D \\
&\qquad = (vu' + M' \otimes M')(B \otimes A)\{(\operatorname{tr} DV)u + \operatorname{vec} MDM'\} \\
&\qquad = (vu' + M' \otimes M')\{(\operatorname{tr} DV)\operatorname{vec} AUB' + \operatorname{vec} AMDM'B'\} \\
&\qquad = (\operatorname{tr} AUB'U)(\operatorname{tr} DV)\operatorname{vec} V + (\operatorname{tr} DV)\operatorname{vec} M'AUB'M \\
&\qquad\quad + (\operatorname{tr} AMDM'B'U)\operatorname{vec} V + \operatorname{vec} M'AMDM'B'M \\
&\qquad = \operatorname{vec}\{(\operatorname{tr} AUB'U)(\operatorname{tr} DV)V + (\operatorname{tr} DV)M'AUB'M \\
&\qquad\quad + (\operatorname{tr} AMDM'B'U)V + M'AMDM'B'M;
\end{aligned}
$$

$$
\begin{aligned}
&(\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \{(vu' + M' \otimes M') \\
&\quad \otimes (vu' + M' \otimes M')\}(C_2^n C_3^n (\operatorname{vec} B \otimes \operatorname{vec} A) \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \{(vu' + M' \otimes M') \otimes (vu' + M' \otimes M')\} \\
&\qquad\quad \cdot (C_2^n (\operatorname{vec} B \otimes \operatorname{vec} A') \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \{(vu' + M' \otimes M') \\
&\qquad\quad \otimes (vu' + M' \otimes M')\} \operatorname{vec}(B \otimes A') \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \operatorname{vec}\{(vu' + M' \otimes M') \\
&\qquad\quad \cdot (B \otimes A')(uv' + M \otimes M)\} \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \operatorname{vec}\{(\operatorname{tr} AUBU)vv' + v(\operatorname{vec} M'AUBM)' \\
&\qquad\quad + (\operatorname{vec} M'A'UB'M)v' + M'BM \otimes M'A'M\} \\
&\qquad = (\operatorname{vec} D \otimes I_{p^2})' C_2^p C_3^p C_2^p \{(\operatorname{tr} AUBU)(v \otimes v) + \operatorname{vec} M'AUBM \otimes v \\
&\qquad\quad + v \otimes \operatorname{vec} M'A'UB'M + \operatorname{vec}(M'BM \otimes M'A'M)\}
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathrm{vec}\, D \otimes I_{p^2})' C_3^p C_2^p C_3^p \{(\mathrm{tr} AUBU)(v \otimes v) + \mathrm{vec}\, M'AUBM \otimes v \\
&\quad + v \otimes \mathrm{vec}\, M'A'UB'M\} + (\mathrm{vec}\, D \otimes I_{p^2})' \\
&\quad \cdot C_2^p C_3^p C_2^p \mathrm{vec}(M'BM \otimes M'A'M) \\
&= (\mathrm{vec}\, D \otimes I_{p^2})' C_3^p C_2^p \{(\mathrm{tr} AUBU)(v \otimes v) + \mathrm{vec}\, M'AUBM \otimes v \\
&\quad + v \otimes \mathrm{vec}\, M'BUAM\} + (\mathrm{vec}\, D \otimes I_{p^2})' \\
&\quad \cdot C_2^p C_3^p (\mathrm{vec}\, M'BM \otimes \mathrm{vec}\, M'A'M) \\
&= (\mathrm{vec}\, D \otimes I_{p^2})' C_3^p \mathrm{vec}\{(\ \mathrm{tr}\, AUBU)(V \otimes V) + M'AUBM \otimes V \\
&\quad + V \otimes M'BUAM\} + (\mathrm{vec}\, D \otimes I_{p^2})' C_2^p (\mathrm{vec}\, M'BM \otimes \mathrm{vec}\, M'AM) \\
&= \mathrm{vec}\{(\mathrm{tr}\, AUBU) K_{pp} \mathrm{vec}\, VDV + K_{pp} \mathrm{vec}\, VDM'B'UA'M \\
&\quad + K_{pp} \mathrm{vec}\, M'BUAMDV\} + (\mathrm{vec}\, D \otimes I_{p^2})' \mathrm{vec}(M'BM \otimes M'AM) \\
&= \mathrm{vec}\{(\mathrm{tr}\, AUBU) VD'V + M'AUBMD'V + VD'M'A'UB'M\} \\
&\quad + \mathrm{vec}\, M'AMDM'B'M;
\end{aligned}
$$

$$
\begin{aligned}
&-2(\mathrm{vec}\, D \otimes I_{p^2})' C_2^p (M \otimes M \otimes M \otimes M)'(\mathrm{vec}\, B \otimes \mathrm{vec}\, A) \\
&\quad = -2(\mathrm{vec}\, D \otimes I_{p^2})' C_2^p (\mathrm{vec}\, M'BM \otimes \mathrm{vec}\, M'AM) \\
&\quad = -2\mathrm{vec}\, M'AMDM'B'M.
\end{aligned}
$$

We used $v := \mathrm{vec}\, V$ and $C_3^p(v \otimes v) = v \otimes v$.

Adding the four terms together and omitting the vec operator yields

$$
\begin{aligned}
&\{(\mathrm{tr}\, AU)V + M'AM\} D \{(\mathrm{tr}\, BU)V + M'B'M\} \\
&\quad + (\mathrm{tr}\, AUB'U)(\mathrm{tr}\, DV)V + (\mathrm{tr}\, DV) M'AUB'M \\
&\quad + (\mathrm{tr}\, AMDM'B'U)V + M'AMDM'B'M \\
&\quad + (\mathrm{tr}\, AUBU)VD'V + M'AUBMD'V \\
&\quad + VD'M'A'UB'M + M'AMDM'B'M - 2M'AMDM'B'M \\
&\quad\quad = \{(\mathrm{tr}\, AU)V + M'AM\} D \{(\mathrm{tr}\, BU)V + M'B'M\} \\
&\quad\quad\quad + (\mathrm{tr}\, AUB'U)(\mathrm{tr}\, DV)V \\
&\quad\quad\quad + (\mathrm{tr}\, DV) M'AUB'M + (\mathrm{tr}\, AMDM'B'U)V \\
&\quad\quad\quad + (\mathrm{tr}\, AUBU)VD'V + VD'M'A'UB'M + M'AUBMD'V. \qquad \square
\end{aligned}
$$

Note 1. Result (9.1) has also been derived by Neudecker and Wansbeek [26] by applying iterated expectations.

Note 2. When $\mathscr{D}(\mathrm{vec} X) = \Phi_{XX} \otimes I_n$, one replaces V by Φ_{XX} and U by I_n in (9.1).

Note 3. It is straightforward to derive $E(S_A D S_B')$ for the general variance $\Omega = \sum_{ij}(E_{ij} \otimes \Psi_{ij})$. As E_{ij} and Ψ_{ij} are generally not symmetric, one should take care when replacing U and V!

10. The covariance of two matrix bilinear forms in two jointly normally distributed matrices

Consider $Z = (X, Y)$, with $(n \times p)X$ and $(n \times q)Y$, both random matrices. Let $\mathrm{vec}Z \sim N\{\mathrm{vec}(M, N), \Omega\}$.

$$\Omega = \begin{bmatrix} \Omega_{XX} & \Omega_{XY} \\ \Omega_{YX} & \Omega_{YY} \end{bmatrix},$$

where $\Omega_{XY} = C(\mathrm{vec}X,\ \mathrm{vec}Y)$.

Define the two matrix bilinear forms

$$S_A := X'AY \quad \text{and} \quad S_B := X'BY.$$

The following result can then be proved:

$$\begin{aligned} &C(\mathrm{vec}\, S_A, \mathrm{vec}\, S_B) \\ &\quad = (I_{pq} \otimes \mathrm{vec}\, A)'(I_q \otimes K_{pn} \otimes I_n)\{\Omega_{YY} \otimes \Omega_{XX} + \Omega_{YY} \otimes \mu\mu' \\ &\qquad + \nu\nu' \otimes \Omega_{XX} + K_{nq,np}(\Omega_{XY} \otimes \Omega_{YX} + \Omega_{XY} \otimes \nu\mu' + \mu\nu' \otimes \Omega_{YX})\} \\ &\qquad \cdot (I_q \otimes K_{np} \otimes I_n)(I_{pq} \otimes \mathrm{vec}\, B), \end{aligned}$$

with $\mu := \mathrm{vec}\, M$ *and* $\nu := \mathrm{vec}\, N$.

Proof. We use the expressions

$$\mathrm{vec}\, S_A = (\mathrm{vec}\, A \otimes I_{pq})'\mathrm{vec}(Y' \otimes X')$$

and

$$\mathrm{vec}\, S_B = (\mathrm{vec}\, B \otimes I_{pq})'\mathrm{vec}(Y' \otimes X').$$

Hence

$$\begin{aligned} &C(\mathrm{vec}\, S_A, \mathrm{vec}\, S_B) \\ &\quad = (\mathrm{vec}\, A \otimes I_{pq})'\mathscr{D}\{\mathrm{vec}(Y' \otimes X')\}(\mathrm{vec}\, B \otimes I_{pq}) \\ &\quad = (\mathrm{vec}\, A \otimes I_{pq})'\mathscr{D}\{K_{n^2,pq}\mathrm{vec}(Y \otimes X)\}(\mathrm{vec}\, B \otimes I_{pq}) \\ &\quad = (\mathrm{vec}\, A \otimes I_{pq})'K_{n^2,pq}\mathscr{D}\{\mathrm{vec}(Y \otimes X)\}K_{pq,n^2}(\mathrm{vec}\, B \otimes I_{pq}) \\ &\quad = (I_{pq} \otimes \mathrm{vec}\, A)'\mathscr{D}\{\mathrm{vec}(Y \otimes X)\}(I_{pq} \otimes \mathrm{vec}\, B) \\ &\quad = (I_{pq} \otimes \mathrm{vec}\, A)'(I_q \otimes K_{pn} \otimes I_n)\{\Omega_{YY} \otimes \Omega_{XX} + \Omega_{YY} \otimes \mu\mu' \\ &\qquad + \nu\nu' \otimes \Omega_{XX} + K_{nq.np}(\Omega_{XY} \otimes \Omega_{YX} + \Omega_{XY} \otimes \nu\mu' + \mu\nu' \otimes \Omega_{YX})\} \\ &\qquad \cdot (I_q \otimes K_{np} \otimes I_n)(I_{pq} \otimes \mathrm{vec}\, B). \end{aligned}$$

We used (5.1). □

Corollaries.

1. $\mathscr{D}(\operatorname{vec} S_A)$

$$= (I_{pq} \otimes \operatorname{vec} A)'(I_q \otimes K_{pn} \otimes I_n)\{\Omega_{YY} \otimes \Omega_{XX} + \Omega_{YY} \otimes \mu\mu' + \nu\nu' \otimes \Omega_{XX} + K_{nq.np}(\Omega_{XY} \otimes \Omega_{YX} + \Omega_{XY} \otimes \nu\mu' + \mu\nu' \otimes \Omega_{YX})\} \cdot (I_q \otimes K_{np} \otimes I_n)(I_{pq} \otimes \operatorname{vec} A).$$

2. $\mathscr{D}(\operatorname{vec} X'AX)$

$$= (I_{p^2} \otimes \operatorname{vec} A)'(I_p \otimes K_{pn} \otimes I_n)\{(I_{n^2p^2} + K_{np.np})(\Omega_{XX} \otimes \Omega_{XX} + \Omega_{XX} \otimes \mu\mu' + \mu\mu' \otimes \Omega_{XX})\}(I_p \otimes K_{np} \otimes I_n)(I_{p^2} \otimes \operatorname{vec} A).$$

For further reading

Additional references to other related work are the following: [1–11, 13–15, 18–21, 23, 24, 29–34].

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References

[1] P. Balestra, A. Holly, A general Kronecker formula for the moments of the multivariate normal distribution, Cahier No. 9002, DEEP, University of Lausanne, Switzerland, 1990.
[2] M.W. Browne, H. Neudecker, The covariance matrix of a general symmetric second degree matrix polynomial under normality assumptions, Linear Algebra Appl. 103 (1988) 13–20.
[3] F.J.H. Don, The expectation of products of quadratic forms in normal variables, Statist. Neerlandica 33 (1979) 73–79.
[4] G.A. Ghazal, Moments of the ratio of two dependent quadratic forms, Statist. Probab. Lett. 20 (1994) 313–319.
[5] G.A. Ghazal, Recurrence formula for expectations of products of quadratic forms, Statist. Probab. Lett. 27 (1996) 101–109.
[6] V. Guiard, A general formula for the central mixed moments of the multivariate normal distribution, Statistics 17 (1986) 279–289.
[7] B. Holmquist, Moments and cumulants from generating functions of Hilbert space valued random variables and an application to the Wishart distribution, Statistical Research Report 3, University of Lund, Sweden, 1985.
[8] B. Holmquist, Moments and cumulants of the multivariate normal distribution, Stochastic Anal. Appl. 6 (1988) 273–278.
[9] K.G. Jinadasa, D.S. Tracy, Higher order moments of random vectors using matrix derivatives, Stochastic Anal. Appl. 4 (1986) 399–407.
[10] T. Kollo, H. Neudecker, Asymptotics of eigenvalues and unit-length eigenvectors of sample variance and correlation matrices, Appendix 1, J. Multivariate Anal. 47 (1993) 283–300.

[11] A. Kumar, Expectation of product of quadratic forms, Sankhya Ser. B 35 (1973) 359–362.
[12] M.A. Legault-Giguère, Multivariate normal estimation with missing data, M.Sc. Thesis, McGill University, Montréal, Québec, Canada, 1974.
[13] J.R. Magnus, The moments of products of quadratic forms in normal variables, Statist. Neerlandica 32 (1978) 201–210.
[14] J.R. Magnus, The expectation of products of quadratic forms in normal variables: the practice, Statist. Neerlandica 33 (1979) 131–136.
[15] J.R. Magnus, The exact moments of a ratio of quadratic forms in normal variables, Ann. Econom. Statist. 4 (1986) 95–109.
[16] J.R. Magnus, H. Neudecker, The commutation matrix: some properties and applications, Ann. Statist. 7 (1979) 381–394.
[17] J.R. Magnus, H. Neudecker, Matrix Differential Calculus with Applications in Statistics and Econometrics, Revised ed., Wiley, Chichester, 1999.
[18] A.M. Mathai, S.B. Provost, Quadratic Forms in Random Variables, Marcel Dekker, New York, 1992.
[19] A. Merckens, T.J. Wansbeek, Formula manipulation in statistics on the computer: evaluating the expectation of higher-degree functions of normally distributed matrices, Comput. Statist. Data Anal. 8 (1989) 189–200.
[20] P.N. Misra, Recurrence formulae for mathematical expectations of products of matrices of structural disturbances, Sankhya Ser. B 34 (1972) 379–384.
[21] H. Neudecker, The Kronecker product and some of its applications in econometrics, Statist. Neerlandica 22 (1968) 69–82.
[22] H. Neudecker, Recent advances in statistical applications of commutation matrices, in: W. Grossman, G. Pflug, I. Vincze, W. Wertz (Eds.), Proceedings of the Fourth Pannonian Symposium on Mathematical Statistics, vol. B, Reidel, Dordrecht, 1985, pp. 239–250.
[23] H. Neudecker, On the dispersion matrix of a matrix quadratic form connected with the noncentral Wishart distribution, Linear Algebra Appl. 70 (1985) 257–262.
[24] H. Neudecker, The variance matrix of a matrix quadratic form under normality assumptions, Statistics 21 (1990) 455–459.
[25] H. Neudecker, T.J. Wansbeek, Some results on commutation matrices with statistical applications, Canad. J. Statist. 11 (1983) 221–231.
[26] H. Neudecker, T.J. Wansbeek, Fourth-order properties of normally distributed random matrices, Linear Algebra Appl. 97 (1998) 13–24.
[27] Shalabh, Improved estimation in measurement error models through Stein rule procedure, J. Multivariate Anal. 67 (1987) 35–48. Corrigendum, J. Multivariate Anal. 74 (2000) 162.
[28] V.K. Srivastava, R. Tiwari, Evaluation of expectations of products of stochastic matrices, Scand. J. Statist. 3 (1976) 135–138.
[29] S.A. Sultan, Moments of multivariate and matrix-variate distributions using matrix derivatives, Ph.D. Thesis, University of Windsor, Ontario, Canada, 1994.
[30] D.S. Tracy, S.A. Sultan, Higher order moments of the multivariate normal distribution using matrix derivatives, Stochastic Anal. Appl. 11 (1993) 337–348.
[31] D. von Rosen, Multivariate linear normal models with special references to the growth curve model, Ph.D. Thesis, University of Stochholm, 1985.
[32] D. von Rosen, Moments for matrix normal variables, Statistics 19 (1988) 575–583.
[33] I. Žežula, Covariance components estimation in the growth curve model, Statistics 24 (1993) 321–330.
[34] Y. Zhao, Covariance matrices of quadratic forms in elliptical distributions, Statist. Probab. Lett. 21 (1994) 131–140.